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Keywords: approximate Caratheodory selections, fixed points of nonconvex valued correspondences, measurable selection valued correspondences, Komlos limits, Komlos' Theorem, weak star convergence, discounted stochastic games, stationary Markov equilibria.

JEL Classification: C7

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Stationary Markov Equilibria for K-Class Discounted Stochastic Games

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Abstract

For a discounted stochastic game with an uncountable state space and compact metric action spaces, we show that if the measurable-selection-valued, *Nash payoff selection correspondence* of the underlying one-shot game contains a sub-correspondence having the K-limit property (i.e., if the Nash payoff selection sub-correspondence contains its K-limits and therefore is a K correspondence), then the discounted stochastic game has a stationary Markov equilibrium. Our key result is a new fixed point theorem for measurable-selection-valued correspondences having the K-limit property. We also show that if the discounted stochastic game is noisy (Duggan, 2012), or if the underlying probability space satisfies the $\mathcal{G}$-nonatomic condition of Rokhlin (1949) and Dynkin and Evstigneev (1976) (and therefore satisfies the coaser transition kernel condition of He and Sun, 2014), then the Nash payoff selection correspondence contains a sub-correspondence having the K-limit property. *Key words and phrases*: approximate Caratheodory selections, fixed points of nonconvex valued correspondences, measurable selection valued correspondences, Komlos limits, Komlos' Theorem, weak star convergence, discounted stochastic games, stationary Markov equilibria.

JEL classification: C7

1 Introduction

For a discounted stochastic game with an uncountable state space and compact metric action spaces, we show that if the measurable-selection-valued, *Nash payoff selection correspondence* of the underlying one-shot game contains a sub-correspondence having the K-limit property (i.e., if the Nash payoff selection sub-correspondence contains its K-limits and therefore is a K correspondence), then the discounted stochastic game has a stationary Markov equilibrium. We will refer to all such discounted stochastic games as K-class discounted stochastic games. Our key result is a new fixed point theorem for measurable-selection-valued correspondences having the K-limit property. The steps in the logic are as follows: First, we show that if the Nash payoff selection correspondence is a K-correspondence, then it is upper semicontinuous and takes nonempty compact values, with respect to the weak star topology, and if in addition the dominating probability measure for the game is nonatomic then it takes nonempty compact and contractible values (i.e., if Nash payoff selection correspondence a K-correspondence and the dominating probability measure is nonatomic, then the Nash payoff selection correspondence is a w^*-w^*-USCO taking contractible values).[1] Because the Nash payoff selection correspondence takes contractible values, it is w^*-w^*-approximable and therefore, has fixed points. According to Blackwell's Theorem (extended to DSGs), in order for the DSG to possess stationary Markov equilibria it is necessary and sufficient that the Nash payoff selection correspondence belonging to the DSG have fixed points. We note that our sufficiency condition (the K limit property) for approximability rules out the key pathology underlying recent existence counter examples due to Levy (2013) and Levy and McLennan (2014). In particular, contractibility rules out Nash equilibria homeomorphic to the unit circle.

We also show that if the discounted stochastic game is noisy (Duggan, 2012), or if the underlying probability space satisfies the $\mathcal{G}$-nonatomic condition of Rokhlin (1949) and Dynkin and Evstigneev (1976) (and therefore satisfies the coaser transition kernel condition of He and Sun, 2014), then the Nash payoff selection correspondence contains a sub-correspondence having the K-limit property. Thus, all noisy and all $\mathcal{G}$-nonatomic discounted stochastic games are K-class.

2 Discounted Stochastic Games

In an m-player, non-zero sum, discounted stochastic game, players seek to choose strategies that will maximize the sum of their discounted future payoffs. The game-theoretic model we will consider here is essentially the model of Nowak and Raghavan (1992). Our objective is to show that if a discounted stochastic game is approximable, then it has stationary Markov equilibria. We begin by listing the primitives and assumptions of the Nowak and Raghavan class of models.

2.1 Primitives and Assumptions

We will use the term correspondence to mean a set-valued mapping, $\Gamma : \Omega \longrightarrow P(X)$, taking nonempty values. Here, $P(X)$ denotes the collection of all nonempty subsets of X. Now to the specifics.

[1] Using methods introduced by Nowak (2003) - i.e., by dividing the state space into a nonatomic part and an atomic part - we can show that if the Nash payoff selection correspondence is contractibly-valued on the nonatomic part, then it is contractibly valued on all of the probability space.

An m-person, non-zero sum discounted stochastic game is defined by the following primitives:

$$DSG := \left\{(\Omega, B_\Omega, \mu), (A_d, \Phi_d(\cdot), r_d(\cdot,\cdot), \beta_d)_{d\in D}, q(\cdot|\cdot,\cdot)\right\}, \tag{1}$$

where DSG satisfies the following list of assumptions [DSG-1]:

(1) D is a finite set of players consisting of $|D| = m$ players,

(2) (Ω, B_Ω, μ) is the state space with typical element ω where Ω is a complete, separable metric (Polish) space with metric ρ_Ω equipped with Borel σ-field B_Ω and probability measure μ;

(3) A_d is the space of actions available to player d with typical element a_d where A_d is a compact, convex subset of a locally convex Hausdorff topological vector space E_d, metrizable - with metric ρ_{A_d} - for the relative topology inherited from E_d;

(4) $\Phi_d(\cdot)$ is the feasible action correspondence, a measurable set-valued mapping from the state space Ω into the nonempty, ρ_{A_d}-compact, convex subsets of A_d with graph

$$Gr\Phi_d(\cdot) := \{(\omega, a_d) \in \Omega \times A_d : a_d \in \Phi_d(\omega)\}. \tag{2}$$

Because $\Phi_d(\cdot)$ is ρ_{A_d}-compact-valued and maps from a separable metric space Ω to a ρ_{A_d}-compact metric space A_d, the measurability of $\Phi_d(\cdot)$ is equivalent to $\Phi_d(\cdot)$ having a measurable graph. Thus, the measurability of $\Phi_d(\cdot)$ is equivalent to $Gr\Phi_d(\cdot) \in B_\Omega \times B_{\rho_{A_d}}$.

Letting $A := \prod_{d\in D} A_d$, equip A with the sum metric,

$$\rho_A := \sum_d \rho_{A_d},$$

a metric compatible with the product topology on A. Thus, A is the ρ_A-compact, convex subset of all possible action profiles in $E := \prod_{d\in D} E_d$ with typical element $a = (a_d, a_{-d}) \in A$. Letting

$$\Phi(\cdot) := \Phi_1(\cdot) \times \cdots \times \Phi_m(\cdot) := \textstyle\prod_{d\in D} \Phi_d(\cdot), \tag{3}$$

$\Phi(\cdot)$ is also a measurable set-valued mapping (Lemma 18.4, Aliprantis-Border, 2006) from the state space Ω into the nonempty, ρ_A-compact, convex subsets of A. In each state $\omega \in \Omega$, $\Phi_d(\omega) \subseteq A_d$ is the ρ_{A_d}-compact, convex subset of feasible actions available to player d in state ω, while $\Phi(\omega) \subseteq A$ is the ρ_A-compact, convex subset of feasible action profiles (m-tuples) available to players in state ω. Letting $Gr\Phi(\cdot)$ denote the graph of $\Phi(\cdot)$, we have

$$Gr\Phi(\cdot) := \{(\omega, a) \in \Omega \times A : a \in \Phi(\omega)\} \in B_\Omega \times B_A. \tag{4}$$

(5) $r_d(\cdot,\cdot)$ is player $d's$ real-valued payoff function defined on $\Omega \times A$, such that for all players $d \in D$ (i) $|r_d(\omega, a)| \leq M$ for all $(\omega, a) \in \Omega \times A$, (ii) $r_d(\cdot, a)$ is measurable on Ω for all $a \in A$, (iii) $r_d(\omega, \cdot)$ is continuous and multilinear on A for all $\omega \in \Omega$;

(6) $q(\cdot|\cdot,\cdot)$ is the law of motion such that (i) for all $(\omega, a) \in \Omega \times A$ the probability measure $q(\cdot|\omega, a)$ defined on (Ω, B_Ω) is absolutely continuous with respect to the probability measure μ defined on (Ω, B_Ω) (i.e., $q(\cdot|\omega, a) << \mu$ for all $(\omega, a) \in \Omega \times A$), (ii) for all sets $E \in B_\Omega$, $q(E|\cdot,\cdot)$ is measurable on $\Omega \times A$, and (iii) the collection of probability density functions

$$H_\mu := \{h(\cdot|\omega, a) : (\omega, a) \in \Omega \times A\}$$

of $q(\cdot|\omega, a)$ with respect to μ is such that for each state $\omega \in \Omega$ the function

$$a := (a_d, a_{-d}) \longrightarrow h(\omega'|\omega, a_d, a_{-d}) \text{ is continuous in } a \text{ and affine in } a_d$$

a.e.$[\mu]$ in ω'.

2.2 Continuation Values

Players in a discounted stochastic game are guided in making their strategy choices by state-contingent prices or values. For each player d, this vector of state-contingent values is given by a function, $v_d : \Omega \longrightarrow R$, and therefore players' state-contingent values are given by a value function profile, $v := (v_1, \ldots, v_m)$. As in the literature on discounted stochastic games (e.g., see Nowak and Raghavan, 1991), the space of players' value function profiles is given by

$$\mathcal{L}^{\infty}_{X} := \mathcal{L}^{\infty}_{X_1} \times \cdots \times \mathcal{L}^{\infty}_{X_m},$$

where for each player $d = 1, 2, \ldots, m$, $\mathcal{L}^{\infty}_{X_d}$ is space of μ-equivalence classes of functions, $v_d : \Omega \longrightarrow R$, such that $v_d(\omega) \in X_d$ a.e. $[\mu]$. For each player d, X_d is the closed bounded interval, $[-M, M]$, the same for each player. Players' payoffs (both immediate and discounted) reside in closed, bounded, convex subset, $X := X_1 \times \cdots \times X_m = [-M, M]^m$, and thus, players' value function profiles reside in the space, $\mathcal{L}^{\infty}_{X}$, a metrizable, weak star compact, convex subset of $\mathcal{L}^{\infty}_{R^m}$.

Formally, let $\mathcal{L}^1_R(\mu, \Omega) := \mathcal{L}^1_R$ denote the separable Banach space of μ-equivalence classes of μ-integrable functions, $u : \Omega \longrightarrow R$ with norm

$$\|u\|_1 := \textstyle\int_{\Omega} |u| \, d\mu.$$

Also, denote by L^1_R the prequotient of $\mathcal{L}^1_R$ (i.e., the space of all real-valued, integrable functions), and let

$$\mathcal{L}^1_{R^m} := \underbrace{\mathcal{L}^1_R \times \cdots \times \mathcal{L}^1_R}_{m \text{ times}}$$

denote the separable Banach space of μ-equivalence classes of μ-integrable functions, $U : \Omega \longrightarrow R^m$, $U := (U_1, \ldots U_d, \ldots, U_m)$, with norm

$$\|U\|_1 = \sum_{d=1}^{m} \|U_d\|_1 .$$

Next, let $\mathcal{L}^{\infty}_R$ denote the Banach space of μ-equivalence classes of μ-essentially bounded functions, $v : \Omega \longrightarrow R$ with norm

$$\|v\|_{\infty} := ess\sup v := \inf \{x \in R : \mu\{\omega : |v(\omega)| > x\} = 0\} .$$

$\mathcal{L}^{\infty}_R$ is the norm dual of $\mathcal{L}^1_R$. Equip $\mathcal{L}^{\infty}_R$ with the weak star topology, denoted by w^* or $\sigma(\mathcal{L}^{\infty}_R, \mathcal{L}^1_R)$. We will denote by L^{∞}_R the prequotient of $\mathcal{L}^{\infty}_R$ (i.e., the space of all real-valued, μ-essentially bounded functions).

For $d = 1, 2, \ldots, m$, let X_d be the closed bounded interval $[-M, M] \subset R$, and let

$$\mathcal{L}^{\infty}_{X_d} := \{v \in \mathcal{L}^{\infty}_R : v(\omega) \in X_d \text{ a.e. } [\mu]\} .$$

Equip $\mathcal{L}^{\infty}_{X_d}$ with the compact and metrizable relative weak star topology, denoted by w^*_d or $\sigma(\mathcal{L}^{\infty}_{X_d}, \mathcal{L}^1_{X_d})$.[2] To fix the metric and hence the notation, let $\rho_{w^*_d}$ be the metric on $\mathcal{L}^{\infty}_{X_d}$

[2] Because the Borel σ-field B_{Ω} is countably generated, the space of μ-equivalence classes of μ-integrable functions, $\mathcal{L}^1_R$, is separable. As a consequence, the set of value function μ-equivalence classes $\mathcal{L}^{\infty}_{X_d}$ is a compact, convex, and metrizable subset of $\mathcal{L}^{\infty}_R$ for the weak star topology (e.g., see Nowak and Raghavan, 1992).

compatible with the weak star topology. Also, let ρ_{X_d} denote the metric on X_d where for x and x' in X_d, $\rho_{X_d}(x, x') := |x - x'|$.

Finally, let $X := X_1 \times \cdots \times X_m$ and consider the Cartesian product,

$$\mathcal{L}^{\infty}_{X} := \mathcal{L}^{\infty}_{X_1} \times \cdots \times \mathcal{L}^{\infty}_{X_m},$$

equipped with the the sum metric,

$$\rho_{w^*} := \textstyle\sum_{d=1}^{m} \rho_{w^*_d},$$

a metric compatible with the relative weak star product topology, w^*, on $\mathcal{L}^{\infty}_{X}$, and equip X with the sum metric

$$\rho_X := \textstyle\sum_{d=1}^{m} \rho_{X_d}.$$

2.3 The Parameterized Collection of One-Shot Games

We know from Blackwell's Theorem (1965) - extended to stochastic games - that in order to find conditions sufficient to guarantee the existence of stationary Markov equilibrium, we must focus on the discounted stochastic game's underlying collection of one-shot games. This collection of one-shot games is parameterized by states and value function profiles. Thus, each value function profile, v, identifies a particular collection of state-contingent, one shot v-games. The crux of the problem is to identify the correct collection of state-contingent v-games for players to play - or more specifically to identify the correct value function profile, say v^*. This problem is a fixed point problem. Our main contribution, therefore, will take the form of a fixed point result for the nonconvex, measurable-selection-valued Nash payoff selection correspondence. Thus, as a consequence of Blackwell's Theorem, our objective will to identify conditions sufficient to guarantee that the Nash payoff selection correspondence, induced from the Nash payoff correspondence, has fixed points. We will then be able to deduce, via our fixed point results, that a correct collection of state-contingent, one-shot v-games exists, and via Blackwell's Theorem, we will be able to conclude that the discounted stochastic game to which this correct collection of state-contingent v-games belongs has a stationary Markov equilibrium.

We begin by discussing a $DSG's$ underlying parameterized collection of one-shot games.

2.3.1 The Ingredients

Given discounted stochastic game,

$$DSG := \left\{(\Omega, B_{\Omega}, \mu), (A_d, \Phi_d(\cdot), r_d(\cdot,\cdot), \beta_d)_{d\in D}, q(\cdot|\cdot,\cdot)\right\},$$

with dominating probability measure, μ, and discount rate profile, $\beta := (\beta_1, \ldots, \beta_m)$, we have for each $(\omega, v) \in \Omega \times \mathcal{L}^{\infty}_{X}$, a one-shot game given by

$$\mathcal{G}(\omega, v) := \{\Phi_d(\omega), U_d(\omega, (\cdot,\cdot), v_d)\}_{d\in D},$$

where for each action choice profile, $a = (a_d, a_{-d}) \in A$, player $d's$ expected one-shot payoff is

$$U_d(\omega, (a_d, a_{-d}), v_d)$$

$$:= (1 - \beta_d) r_d(\omega, (a_d, a_{-d})) + \beta_d \textstyle\int_{\Omega} v_d(\omega') q(d\omega'|\omega, (a_d, a_{-d}))$$

$$= (1 - \beta_d) r_d(\omega, (a_d, a_{-d})) + \beta_d \textstyle\int_{\Omega} v_d(\omega') h(\omega'|\omega, (a_d, a_{-d})) d\mu(\omega').$$

Letting

$$U(\omega, a, v) := (U_1(\omega, a, v_1), \ldots, U_m(\omega, a, v_m)),$$

under assumptions [DSG-1], we can show that in each state, $\omega \in \Omega$, each player's expected payoff function, $(a, v_d) \longrightarrow U_d(\omega, a, v_d) \in X_d$, is $\rho_{A\times w_d^*}$-continuous in $(a, v_d) \in A \times \mathcal{L}^{\infty}_{X_d}$. - so that in each state, $\omega \in \Omega$, the X-valued function,

$$(a, v) \longrightarrow U(\omega, a, v) \in X,$$

is $\rho_{A\times w^*}$-continuous in $(a, v) \in A\times\mathcal{L}^{\infty}_X$ (see the Appendix 1: Mathematical Preliminaries).

A profile of action choices, $a^* \in \Phi(\omega)$, is a Nash equilibrium for the one-shot game, $\mathcal{G}(\omega, v)$, if for each player d

$$U_d(\omega, (a_d^*, a_{-d}^*), v_d) = \max_{a_d \in \Phi_d(\omega)} U_d(\omega, (a_d, a_{-d}^*), v_d).$$

Under assumptions [DSG-1] the one-shot game, $\mathcal{G}(\omega, v)$, always has a nonempty, ρ_A-compact set of Nash equilibria, $\mathcal{N}(\omega, v)$, and using Berge's Maximum Theorem it is straightforward to show that the *Nash correspondence*,

$$\mathcal{N}(\cdot,\cdot) : \Omega \times \mathcal{L}^{\infty}_X \longrightarrow P_f(A)$$

is upper Caratheodory (i.e., $\mathcal{N}(\cdot,\cdot)$ is product measurable in ω and ν and w^*-A-upper semicontinuous in v with nonempty, ρ_A-compact values). Moreover, it is straightforward to show that the *Nash payoff correspondence*,

$$\mathcal{P}(\cdot,\cdot) : \Omega \times \mathcal{L}^{\infty}_X \longrightarrow P_f(X),$$

given by

$$\mathcal{P}(\omega, v) := \{U \in X : U = U(\omega, a, v) \text{ for some } a \in \mathcal{N}(\omega, v)\},$$

is also upper Caratheodory (i.e., $\mathcal{P}(\cdot,\cdot)$ is product measurable in ω and ν and w^*-X-upper semicontinuous in v with nonempty, ρ_X-compact values). We will denote by

$$\mathcal{UC}_{w^*\text{-}A} := \mathcal{UC}(\Omega \times \mathcal{L}^{\infty}_X, P_f(A)) \text{ and } \mathcal{UC}_{w^*\text{-}X} := \mathcal{UC}(\Omega \times \mathcal{L}^{\infty}_X, P_f(X))$$

the collection of all upper Caratheodory correspondences defined on $\Omega\times\mathcal{L}^{\infty}_X$ taking values in $P_f(A)$ and $P_f(X)$ respectively. Thus, under assumptions [DSG-1],

$$\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{w^*\text{-}A} \text{ and } \mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{w^*\text{-}X}.$$

2.3.2 From Action Choices to Strategies

Given a value function profile, $v \in \mathcal{L}^{\infty}_X$, the collection of one-shot games becomes a collection of state-contingent one-shot games,

$$\omega \longrightarrow \mathcal{G}(\omega, v) := \{\Phi_d(\omega), U_d(\omega, (\cdot,\cdot), v_d)\}_{d\in D},$$

with state contingent Nash correspondence, $\omega \longrightarrow \mathcal{N}(\omega, v)$, and state-contingent Nash payoff correspondence, $\omega \longrightarrow \mathcal{P}(\omega, v)$. We can then measurably string together, state-by-state, Nash equilibria to form a profile of Nash equilibrium strategies,

$$\omega \longrightarrow a^*(\omega) := (a_1^*(\omega), \ldots, a_m^*(\omega)).$$

For each ω, $a^*(\omega)$ is a Nash equilibrium for the one-shot v-game, $\mathcal{G}(\omega, v)$, in state ω. Thus, for each player d, the (B_Ω, B_{A_d})-measurable function, $a_d(\cdot) : \Omega \longrightarrow A_d$, is player $d's$ action choice strategy. We will write

$$a^*(\cdot) \in \Sigma(\mathcal{N}(\cdot, v)) := \Sigma(\mathcal{N}_v)$$

to denote that $a^*(\cdot)$ is a Nash equilibrium strategy for the collection of one-shot state-contingent, v-games, $\mathcal{G}(\omega, v)_{\omega\in\Omega}$. Thus, $a^*(\cdot) \in \Sigma(\mathcal{N}_v)$ is an everywhere (B_Ω, B_A)-measurable selection of the v-Nash correspondence, $\omega \longrightarrow \mathcal{N}(\omega, v)$.

2.3.3 Everywhere Nash Payoff Selections

Let $\Sigma(\mathcal{P}(\cdot, v)) := \Sigma(\mathcal{P}_v)$ denote the collection of all (B_Ω, B_X)-measurable selections of the Nash payoff correspondence, $\omega \longrightarrow \mathcal{P}(\omega, v)$. Thus, $U(\cdot) \in \Sigma(\mathcal{P}_v)$ if and only if $U(\omega) \in \mathcal{P}(\omega, v)$ for all ω. By the Measurable Implicit Function Theorem (Himmelberg, 1975, Theorem 7.1), for $U(\cdot) \in \Sigma(\mathcal{P}_v)$, there exists $a^*(\cdot) \in \Sigma(\mathcal{N}_v)$ such that $U(\omega) = U(\omega, a^*(\omega), v)$ for all ω and $U(\cdot, a^*(\cdot), v) \in \Sigma(\mathcal{P}_v)$. Conversely, if $a^*(\cdot) \in \Sigma(\mathcal{N}_v)$, then $U(\cdot, a^*(\cdot), v) \in \Sigma(\mathcal{P}_v)$.

2.3.4 Payoffs and Probabilities under Stationary Markov Strategies

A stationary Markov strategy for player d, is a (B_Ω, B_{A_d})-measurable function, $a_d(\cdot) : \Omega \longrightarrow A_d$, such that $a_d(\omega) \in \Phi_d(\omega)$ for all ω. Thus, the collection of all player d stationary Markov strategies is given by $\Sigma(\Phi_d)$, the collection of all (everywhere) measurable selections of $\Phi_d(\cdot)$.[3] A Markov strategy profile is given by,

$$(a_1(\cdot), \ldots, a_m(\cdot)) \in \Sigma(\Phi),$$

where

$$\Sigma(\Phi) := \prod_{d \in D} \Sigma(\Phi_d)$$

is the collection of all such profiles.

Let

$$r_d^n(a(\cdot))(\omega) := \begin{cases} r_d(\omega, a(\omega)) & \text{for } n = 1 \\ \int_\Omega r_d(\omega', a(\omega'))q^{n-1}(\omega'|\omega, a(\omega)) & \text{for } n \geq 2, \end{cases} \tag{5}$$

denote the n^{th} period *expected* payoff to player d under Markov strategy profile $a(\cdot)$ starting at state ω given law of motion $q(\cdot|\cdot, \cdot)$. Here, for $n \geq 2$, $q^n(\cdot|\omega, a(\omega))$ is defined recursively by

$$\left.\begin{array}{l} q^n(E|\omega, a(\omega)) \\ = \int_\Omega q(E|\omega', a(\omega'))q^{n-1}(\omega'|\omega, a(\omega)). \end{array}\right\} \tag{6}$$

The discounted expected payoff to player d, with discount rate $\beta_d \in [0, 1)$, over an infinite time horizon under Markov strategy profile $a(\cdot)$ starting at state ω is given by

$$Er_d^\infty(a(\cdot))(\omega) := \textstyle\sum_{n=1}^\infty \beta_d^{n-1} Er_d^n(a(\cdot))(\omega). \tag{7}$$

A stationary Markov strategy profile $a^*(\cdot) \in \Sigma(\Phi)$ is a *stationary Markov equilibrium* if for all players d and in all states ω,

$$Er_d^\infty(a_d^*(\cdot), a_{-d}^*(\cdot))(\omega) \geq Er_d^\infty(a_d'(\cdot), a_{-d}^*(\cdot))(\omega),$$

for all other strategies, $a_d'(\cdot) \in \Sigma(\Phi_d)$.

[3] Thus, $a_d(\cdot) \in \Sigma(\Phi_d)$ if and only if

$$a_d(\cdot) : \Omega \longrightarrow A$$

is (B_Ω, B_{A_d})-measurable and $a_d(\omega) \in \Phi_d(\omega)$ for all ω. Such a strategy is stationary because it does not depend on time (the same strategy applies at all time points). Such a strategy is Markov because the action choice specified by the strategy in a function of the current state - and nothing else.

2.4 Other Continuity Properties

In the underlying one-shot game, each player's expected payoff function, $U_d(\cdot,\cdot,\cdot)$ for $d = 1, 2, \ldots, m$, is given by,

$$U_d(\omega, a, v_d) := \int_\Omega \underbrace{[(1-\beta_d)r_d(\omega, a) + \beta_d v_d(\omega')h(\omega'|\omega, a)]}_{u_d(\omega,a,v_d(\omega'))} d\mu(\omega'). \tag{8}$$

Let

$$U(\omega, a, v) := (U_1(\omega, a, v_1), \ldots, U_m(\omega, a, v_m))$$
and
$$u(\omega, a, v(\omega')) := (u_1(\omega, a, v_1(\omega')), \ldots, u_m(\omega, a, v_m(\omega'))).$$

(1) By part (iii) of assumption (6) we have via Scheffee's Theorem (see Billingsley, 1986, Theorem 16.11) that

$$\left.\begin{array}{l} \sup_{E\in\mathcal{B}(\Omega)} |q(E|\omega, a^n) - q(E|\omega, a^*)|_R \\ \leq \int_\Omega |h(\omega'|\omega, a^n) - h(\omega'|\omega, a^*)|_R \, d\mu(\omega') \to 0, \end{array}\right\} \tag{9}$$

for any sequence of action profiles $\{a^n\}_n$ in $\Phi(\omega)$ converging to $a^* \in \Phi(\omega)$. Thus, $a^n \underset{\rho_A}{\longrightarrow} a^*$ implies that

$$\sup_{E\in\mathcal{B}(\Omega)} |q(E|\omega, a^n) - q(E|\omega, a^*)|_R \longrightarrow 0,$$

sometimes written $\|q(\cdot|\omega, a^n)) - q(\cdot|\omega, a^*)\|_\infty \longrightarrow 0$.

(2) As noted above, under assumptions (5) and (6), in each state, $\omega \in \Omega$, each player's expected payoff function, $(a, v_d) \longrightarrow U_d(\omega, a, v_d) \in X_d$, is $\rho_{A\times w_d^*}$-continuous in $(a, v_d) \in A \times \mathcal{L}^\infty_{X_d}$ - so that in each state, $\omega \in \Omega$, the X-valued function,

$$(a, v) \longrightarrow U(\omega, a, v) \in X,$$

is $\rho_{A\times w^*}$-continuous in $(a, v) \in A \times \mathcal{L}^\infty_X$. In fact, we can say more about the collection of functions, $U(\omega, \cdot, v) : A \longrightarrow X$, for $(\omega, v) \in \Omega \times \mathcal{L}^\infty_X$. In particular, *for each state* $\omega \in \Omega$ *the collection of functions,*

$$\{U(\omega, \cdot, v) : v \in \mathcal{L}^\infty_X\},$$

is uniformly equicontinuous on $\Phi(\omega)$.[4] To see this, let

$$U_{\omega v_d}(\cdot) := (1-\beta_d)r_d(\omega, \cdot) + \beta_d \textstyle\int_\Omega v_d(\omega')h(\omega'|\omega, \cdot))d\mu(\omega').$$

For fixed ω, we have for each $v \in \mathcal{L}^\infty_X$

$$\begin{array}{l} |U_{\omega v_d}(a) - U_{\omega v_d}(a')| \\ \leq (1-\beta_d)\,|r_d(\omega, a) - r_d(\omega, a')| \\ +\beta_d M \left|\int_\Omega h(\omega'|\omega, a)d\mu(\omega') - \int_\Omega h(\omega'|\omega, a')d\mu(\omega')\right|. \end{array}$$

Because

$$r_d(\omega, \cdot) \text{ and } H_\omega(a) := \int_\Omega h(\omega'|\omega, \cdot)d\mu(\omega')$$

[4] The collection, $\{U(\omega, \cdot, v) : v \in \mathcal{L}^\infty_X\}$, is uniformly equicontinuous if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any a and a' in $\Phi(\omega)$ with $\rho_A(a, a') < \delta$,

$$\rho_X(U(\omega, a, v), U(\omega, a', v)) < \varepsilon,$$

for all $v \in \mathcal{L}^\infty_X$.

are continuous functions on a compact set, and hence uniformly continuous, for any $\frac{\varepsilon}{2} > 0$ there is a $\delta > 0$ such that for any a and a' in $\Phi(\omega)$ with $d_A(a, a') < \delta$

$$|r_d(\omega, a) - r_d(\omega, a'))| < \tfrac{\varepsilon}{2}$$
$$\text{and}$$
$$|H_\omega(a) - H_\omega(a')| < \tfrac{\varepsilon}{2}.$$

2.5 Nash Payoff Selections

2.5.1 The Definition

A Nash payoff selection is a function, $U_{(\cdot)} \in \mathcal{L}_X^\infty$ such that $U_\omega \in \mathcal{P}(\omega, v)$ a.e. $[\mu]$ for some fixed value function profile, $v \in \mathcal{L}_X^\infty$. Given parameterized games, $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}_X^\infty}$, satisfying assumptions [DSG-1] with Nash payoff correspondence, $\mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{w^*\text{-}X}$, the induced *Nash payoff selection correspondence* is given by

$$v \longrightarrow S^\infty(\mathcal{P}_v) := \left\{U_{(\cdot)} \in \mathcal{L}_X^\infty : U_\omega \in \mathcal{P}(\omega, v) \text{ a.e. } [\mu]\right\}.$$

Thus, for each value function profile v, $\mathcal{S}^\infty(\mathcal{P}_v)$ is the set of all μ-equivalence classes of measurable selections of the measurable correspondence,

$$\omega \longrightarrow \mathcal{P}_v(\omega) := \mathcal{P}(\omega, v),$$

Recall that $\Sigma^\infty(\mathcal{P}_v)$ denotes the prequotient of $S^\infty(\mathcal{P}_v)$ while $\Sigma(\mathcal{P}_v)$ denotes the set of all everywhere measurable selections of $\mathcal{P}_v(\cdot)$ (for a given v).[5] Because the Nash payoff correspondence, $\omega \longrightarrow \mathcal{P}_v(\omega)$, is (B_Ω, B_X)-measurable with nonempty compact values in X, by the Kuratowski-Ryll-Nardzewski Selection Theorem (1965), $\mathcal{P}_v(\cdot)$ has (B_Ω, B_X)-measurable selections (i.e., $S^\infty(\mathcal{P}_v) \neq \varnothing$).

2.5.2 Decomposability

In general, a subset $\mathcal{S}$ of $\mathcal{L}_X^\infty$ is said to be decomposable if for any two functions $U^0_{(\cdot)}$ and $U^1_{(\cdot)}$ in $\mathcal{S}$ and for any $E \in B_\Omega$, we have

$$U^0_{(\cdot)} I_E(\cdot) + U^1_{(\cdot)} I_{\Omega\backslash E}(\cdot) \in \mathcal{S}.$$

For the Nash payoff correspondence, $\mathcal{P}(\cdot,\cdot) : \Omega \times \mathcal{L}_X^\infty \longrightarrow P_f(X)$, an upper Caratheodory correspondence, for each $v \in \mathcal{L}_X^\infty$, the induced Nash payoff selection correspondence, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, takes decomposable values. Moreover, for each v, $\mathcal{S}^\infty(\Gamma_v)$ is $\|\cdot\|_1$-closed (or $\mathcal{L}^1_{R^m}$-closed) in $\mathcal{L}^\infty_{R^m}$. Thus, for any sequence $\{U^n_{(\cdot)}\}_n$ in $\mathcal{S}^\infty(\mathcal{P}_v)$ converging in $\mathcal{L}^1_{R^m}$-norm to $U^0_{(\cdot)} \in \mathcal{L}^\infty_{R^m}$, we have $U^0_{(\cdot)} \in \mathcal{S}^\infty(\mathcal{P}_v)$. We will denote by $cl_1\mathcal{S}^\infty(\mathcal{P}_v)$ the $\mathcal{L}^1_{R^m}$-closure of $\mathcal{S}^\infty(\mathcal{P}_v)$ in $\mathcal{L}^\infty_{R^m}$. By Lemma 1 in Pales and Zeidan (1999), we know that, in addition to $\mathcal{S}^\infty(\mathcal{P}_v)$ being decomposable, $\mathcal{S}^\infty(\mathcal{P}_v)$ is $\mathcal{L}^1_{R^m}$-closed in $\mathcal{L}^\infty_{R^m}$. Thus, we have

$$cl_1\mathcal{S}^\infty(\mathcal{P}_v) = \mathcal{S}^\infty(\mathcal{P}_v).$$

We also know by Corollary 1 in Pales and Zeidan (1999) that

$$cl_1\mathcal{S}^\infty(\mathcal{P}_v) = \left\{U_{(\cdot)} \in \mathcal{L}^\infty_{R^m} : \exists\ \{U^n_{(\cdot)}\}_n \subset \mathcal{S}^\infty(\mathcal{P}_v) \text{ such that } \lim_n \left\|U^n_{(\cdot)} - U_{(\cdot)}\right\|_1 = 0\right\}.$$

Finally, note that $\mathcal{L}_X^\infty$ is $\mathcal{L}^1_{R^m}$-closed in $\mathcal{L}^\infty_{R^m}$ and decomposable.

[5] A (B_Ω, B_X)-measurable function, $U_{(\cdot)}$, is an everywhere measurable selection of $\mathcal{P}_v(\cdot)$ provided $U_\omega \in \mathcal{P}_v(\omega)$ for all $\omega \in \Omega$.

2.5.3 Sequences of Nash Payoff Selections and Sequences of Nash Equilibria

Consider a sequence,

$$\left\{(v^n, U^n_{(\cdot)}\right\}_n \subset GrS^{\infty}(\mathcal{P}_{(\cdot)}) \subset \mathcal{L}^{\infty}_X \times \mathcal{L}^{\infty}_X,$$

where for each n, $U^n_{(\cdot)} \in \mathcal{L}^{\infty}_X$ is a Nash payoff selection, that is,

$$U^n_{\omega} \in \mathcal{P}(\omega, v^n) \text{ a.e. } [\mu].$$

Let N^{∞} be the exceptional set (i.e., the set of μ-measure zero) such for $\omega \in \Omega \backslash N^{\infty}$, $U^n_{\omega} \in \mathcal{P}(\omega, v^n)$ for all n.

For each n, we have by the Measurable Implicit Function Theorem (e.g., Himmelberg, 1975, Theorem 7.1) a (B_{Ω}, B_A)-measurable function, $a^n(\cdot) : \Omega \longrightarrow A$, such that for each n and $\omega \in \Omega \backslash N^{\infty}$, $a^n(\omega) \in \mathcal{N}(\omega, v^n)$. Thus, we have for each n and $\omega \in \Omega \backslash N^{\infty}$,

$$U^n_{\omega} = U(\omega, a^n(\omega), v^n) \text{ for all } \omega \in \Omega \backslash N^{\infty}, \tag{10}$$

where

$$U(\omega, a^n(\omega), v^n) := (U_1(\omega, a^n(\omega), v^n_1), \ldots, U_m(\omega, a^n(\omega), v^n_m)).$$

Note that under assumptions [DSG-1], the sequence $\{U^n_{(\cdot)}\}_n \subset \mathcal{L}^1_{R^m}$ is $\|\cdot\|_1$-bounded.

2.6 The Problem

As we have discussed in the introduction, by Blackwell's Theorem (1965) we know that a stationary Markov strategy profile,

$$a^*(\cdot) := (a^*_1(\cdot), \ldots, a^*_m(\cdot)) \in \Sigma(\mathcal{N}_{v^*}),$$

is a Nash equilibrium of a discounted stochastic game if and only if there exists a profile of continuation values (or value functions), $v^* := (v^*_1, \ldots, v^*_m) \in \mathcal{L}^{\infty}_X$ such that $v^*(\omega) \in \mathcal{P}(\omega, v^*)$ for all ω, i.e., such that,

$$v^*(\cdot) := (v^*_1(\cdot), \ldots, v^*_m(\cdot)) \in \Sigma(\mathcal{P}_{v^*}),$$

and such that together the pair, $(a^*(\cdot), v^*(\cdot)) \in \Sigma(\mathcal{N}_{v^*}) \times \Sigma(\mathcal{P}_{v^*})$. Equivalently, $a^*(\cdot)$ is a stationary Markov equilibrium if and only if the pair, $(a^*(\cdot), v^*(\cdot))$, satisfy the following system of equations:

for players $d = 1, 2, \ldots, m$ and for all initial states ω

$$v^*_d(\omega) = \underbrace{(1 - \beta_d) r_d(\omega, a^*(\omega)) + \beta_d \int_{\Omega} v^*_d(\omega') h(\omega' | \omega, a^*(\omega)) d\mu(\omega')}_{U_d(\omega, a^*(\omega), v^*_d)} \tag{11}$$

and

$$U_d(\omega, a^*_d(\omega), a^*_{-d}(\omega), v^*_d) = \max\nolimits_{a \in \Phi_d(\omega)} U_d(\omega, a, a^*_{-d}(\omega), v^*_d). \tag{12}$$

Thus, if for the given strategy profile, $a^*(\cdot)$, $v^*(\cdot)$, satisfies state-by-state for each player d the Bellman equations (11), and if for the given value function profile, $v^*(\cdot)$, $a^*(\cdot)$, satisfies state-by-state for each player d the Nash conditions (12), then together, $(a^*(\cdot), v^*(\cdot))$, satisfy Blackwell's conditions, and by Blackwell's Theorem, $a^*(\cdot)$ is a stationary Markov equilibrium of the discounted stochastic game with underlying state-contingent, collection of one-shot games, $\{\mathcal{G}(\omega, v^*)\}_{\omega \in \Omega}$.

Note that if $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$ is the Nash equilibria correspondence for the one-shot game, $(\omega, v) \longrightarrow \mathcal{G}(\omega, v)$, and if $(\omega, v) \longrightarrow \mathcal{P}(\omega, v)$ is the induced Nash equilibria payoff correspondence given by

$$\mathcal{P}(\omega, v) := \{U \in X : U_d = U_d(\omega, a, v_d) \forall d \text{ and some } a \in \mathcal{N}(\omega, v)\}$$

then by Blackwell's Theorem (1965) the discounted stochastic game with underlying collection of one-shot games,

$$\mathcal{G}(\omega, v)_{(\omega,v)\in\Omega\times\mathcal{L}_X^\infty},$$

has a stationary Markov equilibrium if and only if there is a value function profile, $\overline{v}^*$, such that

$$\overline{v}^*(\omega) \in \mathcal{P}(\omega, \overline{v}^*) \text{ a.e. } [\mu],$$

or equivalently, if and only if there is a value function profile, $\overline{v}^*$, such that

$$\overline{v}^* \in \mathcal{S}^\infty(\mathcal{P}(\cdot, \overline{v}^*)),$$

where for each v, $\mathcal{S}^\infty(\mathcal{P}(\cdot, v))$ is the set of μ-equivalence classes of measurable selections of the Nash payoff correspondence, $\omega \longrightarrow \mathcal{P}(\omega, v)$. Once we have found a fixed point,

$$\overline{v}^* \in \mathcal{S}^\infty(\mathcal{P}_{\overline{v}^*}) := \mathcal{S}^\infty(\mathcal{P}(\cdot, \overline{v}^*)),$$

or equivalently a solution to the Bellman inclusion and in particular, a $\overline{v}^* \in \mathcal{L}_X^\infty$ such that

$$\overline{v}^*(\omega) \in \mathcal{P}(\omega, \overline{v}^*) \text{ a.e. } [\mu],$$

we can easily deduce the existence of an everywhere measurable selection $v^* \in \Sigma(\mathcal{P}(\cdot, v^*))$ such that $v^* = \overline{v}^*$ a.e. $[\mu]$ and from this we can easily deduce the existence of the strategy profile, $a^*(\cdot)$, such that $a^*(\cdot) \in \Sigma(\mathcal{N}(\cdot, v^*))$ using the Measurable Implicit Function Theorem (e.g., Himmelberg, 1975, Theorem 7.1). Thus, in order to establish the existence of a stationary Markov equilibrium for our discounted stochastic game it follows from Blackwell's Theorem (1965) that it is both necessary and sufficient that there exists a fixed point, v^*, of the corresponding the Nash payoff selection correspondence, $v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v))$ or equivalently, that the Bellman inclusion have a solution. Formally, we have the following variation on Blackwell's Theorem (1965):

Theorem 1 *(Necessary and sufficient conditions for the existence of stationary Markov equilibria):*
Let

$$DSG := \left\{(\Omega, B_\Omega, \mu), (A_d, \Phi_d(\cdot), r_d(\cdot,\cdot), \beta_d)_{d\in D}, q(\cdot|\cdot,\cdot)\right\},$$

be a discounted stochastic game satisfying assumptions [DSG-1], with Nash payoff correspondence, $\mathcal{P}(\cdot,\cdot)$, for the underlying one-shot game. Then DSG has a stationary Markov equilibrium if and only if the Nash payoff selection correspondence,

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v)),$$

has a fixed point.

As a consequence of Theorem 1 above, our sufficient conditions for existence will take the form of a new fixed point theorem for nonconvex, measurable-selection-valued correspondences.

3 K-Class Discounted Stochastic Games

While the Nash payoff selection correspondence,

$$v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v),$$

takes nonempty, decomposable, $\mathcal{L}^1_X$-norm closed values in $\mathcal{L}^{\infty}_X$ (see Pales and Zeidan, 1999). In general, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is neither w^*-closed valued nor convex-valued. This makes the fixed point problem we must solve in order to establish existence very difficult. However, we will show that if the Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, contains a sub-correspondence, $s^{\infty}(\cdot)$, having the K-limit property (i.e., if $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ contains a sub-correspondence $s^{\infty}(\cdot)$ that is a K-correspondence), then this sub-correspondence, $s^{\infty}(\cdot)$, is a w^*-w^*-USCO and if, in addition, the dominating probability measure μ is nonatomic, then $s^{\infty}(\cdot)$ is a w^*-w^*-USCO taking contractible values. By dividing the state space into a nonatomic part and an atomic part, we can then show (using methods introduced by Nowak, 2003) that if the Nash payoff selection correspondence is contractibly-valued on the nonatomic part, it is contractibly valued on all of the probability space. Due to contractibility, we will be able to conclude via results due to Gorniewicz, Granas, and Kryszewski (1991) that $s^{\infty}(\cdot)$, is w^*-w^*-approximable and therefore has fixed points.

3.1 Approximable Sub-Correspondences

Let $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^{\infty}_X}$ be the collection of one-shot games underlying discounted stochastic game, DSG, satisfying [DSG-1] having Nash correspondence, $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{w^*\text{-}A}$, and Nash payoff correspondence, $\mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{w^*\text{-}X}$.

Consider the induced *Nash payoff selection correspondence*,

$$v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v) := \left\{U_{(\cdot)} \in \mathcal{L}^{\infty}_X : U_{\omega} \in \mathcal{P}(\omega, v) \text{ a.e. } [\mu]\right\}.$$

The Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, is in general not a w^*-w^*-USCO, nor is it convex valued. But it may contain a sub-correspondence that is a w^*-w^*-USCO - and not only that, but $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ may contain a sub-correspondence which is contractibly valued. Are there conditions sufficient to guarantee that $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ contains a contractibly valued w^*-w^*-sub-USCO?

We say that a correspondence

$$s^{\infty}(\cdot) : \mathcal{L}^{\infty}_X \longrightarrow P(\mathcal{L}^{\infty}_X)$$

is a sub-correspondence belonging to $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, denoted by

$$s^{\infty}(\cdot) \in \mathcal{SC}[\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})],$$

if

$$Grs^{\infty}(\cdot) \subseteq Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}),$$

where

$$Grs^{\infty}(\cdot) := \left\{(v, U_{(\cdot)}) \in \mathcal{L}^{\infty}_X \times \mathcal{L}^{\infty}_X : U_{(\cdot)} \in s^{\infty}(v)\right\},$$

and

$$Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) := \left\{(v, U_{(\cdot)}) \in \mathcal{L}^{\infty}_X \times \mathcal{L}^{\infty}_X : U_{(\cdot)} \in \mathcal{S}^{\infty}(\mathcal{P}_v)\right\}.$$

Definition 1 *(Approximable Sub-Correspondences)*
We say that a Nash payoff selection sub-correspondence,

$$s^{\infty}(\cdot) \in \mathcal{SC}[\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})],$$

is w^*-w^*-approximable if for each $\varepsilon > 0$, there exists a w^*-w^*-continuous function,

$$g^{\varepsilon}(\cdot) : \mathcal{L}_X^{\infty} \longrightarrow \mathcal{L}_X^{\infty},$$

such that for each $(v, U_{(\cdot)}) \in Grg^{\varepsilon} \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$ (i.e., for each $(v, U_{(\cdot)}) \in \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$, with $U_{(\cdot)} = g^{\varepsilon}(v) \in \mathcal{L}_X^{\infty}$) there exists

$$(\overline{v}, \overline{U}_{(\cdot)}) \in Grs^{\infty}(\cdot) \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$$

(i.e., there exists $\overline{U}_{(\cdot)} \in \ s^{\infty}(\overline{v})$) such that

$$\rho_{w^*}(v, \overline{v}) + \rho_{w^*}(U_{(\cdot)}, \overline{U}_{(\cdot)}) < \varepsilon. \tag{13}$$

Equivalently, for any $\varepsilon > 0$

$$Grg^{\varepsilon} \subset B_{w^* \times w^*}(\varepsilon, Grs^{\infty}(\cdot)).$$

Thus, the graph of the continuous function $g^{\varepsilon} : \mathcal{L}_X^{\infty} \longrightarrow \mathcal{L}_X^{\infty}$ is contained in the $\rho_{w^* \times w^*}$-open ball of radius ε about the graph of $s^{\infty}(\cdot)$. If $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ contains an w^*-w^*-approximable sub-USCO, $s(\cdot)$, then $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, is w^*-w^*-approximable.

3.2 K-Class Games

We have the following formal definition of the K-limit property.

Definition 2 *(The K-Limit Property and K-Correspondences):*
The Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, has the K-limit property or is a K-correspondence if there exists a sub-correspondence, $s^{\infty}(\cdot) \in \mathcal{SC}[\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})]$, such that for any K-converging sequence,

$$\{(v^n, U^n_{(\cdot)})\}_n \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty},$$

with $U^n_{(\cdot)} \in s^{\infty}(v^n)$ for all n and K-limit, $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$,

$$\widehat{U}_{(\cdot)} \in s^{\infty}(\widehat{v}).$$

If the graph of the Nash payoff selection correspondence, $Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$, contains a K-closed subset,

$$s^{\infty} \subset Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$$

such that for all $v \in \mathcal{L}_X^{\infty}$,

$$s^{\infty}(v) := \left\{U_{(\cdot)} \in \mathcal{L}_X^{\infty} : (v, U_{(\cdot)}) \in s^{\infty}\right\} \neq \varnothing,$$

then the induced mapping, $v \longrightarrow s^{\infty}(v)$, is a w^*-w^*-sub-USCO belonging to $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$.

Given sub-correspondence, $s^{\infty}(\cdot) \in \mathcal{SC}[\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})]$, consider a sequence,

$$\left\{(v^n, U^n_{(\cdot)}\right\}_n \subset Grs^{\infty}(\cdot) \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty},$$

where for each n, $U^n_{(\cdot)} \in \mathcal{L}_X^{\infty}$ is a Nash payoff selection, that is,

$$U^n_{\omega} \in \mathcal{P}(\omega, v^n) \text{ a.e. } [\mu].$$

Let N^∞ be the exceptional set (i.e., the set of μ-measure zero) such for $\omega \in \Omega \backslash N^\infty$, $U_\omega^n \in \mathcal{P}(\omega, v^n)$ for all n. For each n, we have by the Measurable Implicit Function Theorem (e.g., Himmelberg, 1975, Theorem 7.1) a (B_Ω, B_A)-measurable function, $a^n(\cdot) : \Omega \longrightarrow A$, such that for each n and $\omega \in \Omega \backslash N^\infty$, $a^n(\omega) \in \mathcal{N}(\omega, v^n)$. Thus, we have for each n and $\omega \in \Omega \backslash N^\infty$,

$$U_\omega^n = U(\omega, a^n(\omega), v^n) \in \mathcal{P}(\omega, v^n) \text{ a.e. } [\mu],$$

and in fact, we have

$$\{(v^n, U(\cdot, a^n(\cdot), v^n)\}_n \subset Grs^\infty(\cdot) \subset \mathcal{L}_X^\infty \times \mathcal{L}_X^\infty.$$

an alternative statement of the K-limit property is

$\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, has the K-limit property or is a K-correspondence if there exists a sub-correspondence, $s^\infty(\cdot) \in \mathcal{SC}[\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})]$, such that for any K-converging sequence,

$$\{(v^n, U(\cdot, a^n(\cdot), v^n))\}_n \subset \mathcal{L}_X^\infty \times \mathcal{L}_X^\infty,$$

with $U(\cdot, a^n(\cdot), v^n) \in s^\infty(v^n)$ for all n and K-limit, $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}_X^\infty \times \mathcal{L}_X^\infty$,

$$\widehat{U}_\omega \in U(\omega, Ls\{a^n(\omega), \widehat{v}) \text{ a.e. } [\mu],$$

where

$$U(\omega, Ls\{a^n(\omega), \widehat{v}) := \{U(\omega, a, \widehat{v}) \in X : a \in Ls\{a^n(\omega)\}\}.$$

If the parameterized collection of one-shot games, $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}_X^\infty}$, belonging to a DSG satisfying [DsG-1] has a Nash payoff selection correspondence having the K-limit property we will refer to the DSG as being K-class.

3.3 A Fixed Point Theorem for K-Class Discounted Stochastic Games

In this section, we will show that if the Nash payoff selection correspondence, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, has a sub-correspondence,

$$s^\infty(\cdot) \in \mathcal{SC}[\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})]$$

that is a K-correspondence, then this sub-correspondence is in fact a w^*-w^*-sub-USCO belonging to $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$. Moreover, if the probability measure, μ, on the state space, Ω, is nonatomic, then $s^\infty(\cdot)$ takes contractible values. In fact, by dividing the state space into a nonatomic part and an atomic part, we can, using methods introduced by Nowak (2003), show that if the Nash payoff selection correspondence is contractibly-valued on the nonatomic part, then it is contractibly valued on all of the probability space. Thus, the K limit property alone is sufficient to show that a sub-correspondence, $s^\infty(\cdot)$, belonging to $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is a contractibly-valued sub-USCO. But here we will explicitly assume that μ is nonatomic and provide a proof that $s^\infty(\cdot)$ is contractibly-valued with respect to the metric ρ_{w^*} on $\mathcal{L}_X^\infty$ - a metric compatible the relative weak star topology on $\mathcal{L}_X^\infty$.

Theorem 2 *(Any sub-correspondence, $s(\cdot)$, belonging to $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ having the K-limit property is a contractibly-valued USCO)*
Let $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}_X^\infty}$ be the parameterized one-shot game underlying a discounted stochastic game satisfying assumptions [DSG-1] and let

$$\mathcal{S}^\infty(\mathcal{P}_{(\cdot)}) : \mathcal{L}_X^\infty \longrightarrow P(\mathcal{L}_X^\infty),$$

be the Nash payoff selection correspondence. If

$$s^{\infty}(\cdot) \in \mathcal{SC}[\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})]$$

is a K-correspondence (i.e., has the K-limit property), then the following statements are true:
(1) The sub-correspondence, $v \longrightarrow s^{\infty}(v)$, is a w^*-w^*-USCO belonging to $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, that is,

$$s^{\infty}(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}[\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})].$$

(2) For $s^{\infty}(\cdot) \in \mathcal{SC}[\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})]$, if the underlying probability measure, μ, is nonatomic, then for each $v \in \mathcal{L}^{\infty}_X$, $s^{\infty}(v)$ is contractible (with respect to the w^*topology).

PROOF: (1) Because $s^{\infty}(\cdot)$ has the K-limit property, it follows from Komlos Theorem and Theorem A2.3(1) that for each $v \in \mathcal{L}^{\infty}_X$, $s^{\infty}(v)$ is w^*-compact. Therefore, to show that $s^{\infty}(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$, it suffices to show that $Grs^{\infty}(\cdot)$ is $\rho_{w^* \times w^*}$-closed in $\mathcal{L}^{\infty}_X \times \mathcal{L}^{\infty}_X$. Let $\{(v^n, U^n_{(\cdot)})\}_n$ be any sequence in $Grs^{\infty}(\cdot)$ such that

$$v^n \underset{K}{\longrightarrow} \widehat{v} \text{ and } U^n_{(\cdot)} \underset{K}{\longrightarrow} \widehat{U}_{(\cdot)}.$$

Thus, $\{(v^n, U^n_{(\cdot)})\}_n$ is a sequence of payoff selections (rather than a sequence of payoff graph selections). By Theorem A2.3(1), we have $v^n \underset{w^*}{\longrightarrow} v^*$ and $U^n_{(\cdot)} \underset{w^*}{\longrightarrow} U^*_{(\cdot)}$ with $v^* = \widehat{v}$ and $\widehat{U}_{(\cdot)} = U^*_{(\cdot)}$ a.e. $[\mu]$. Also, we have for each n,

$$U^n_{\omega} \in \mathcal{P}(\omega, v^n) \text{ a.e. } [\mu].$$

By the K-limit property of $s^{\infty}(\cdot)$, we have that

$$\widehat{U}_{(\cdot)} \in s^{\infty}(\widehat{v}),$$

and given that $\widehat{U}_{(\cdot)} = U^*_{(\cdot)}$ a.e. $[\mu]$, we have

$$U^*_{(\cdot)} \in s^{\infty}(\widehat{v}).$$

Given that $v^* = \widehat{v}$ a.e. $[\mu]$, we have $\mathcal{P}(\omega, v^*) = \mathcal{P}(\omega, \widehat{v})$ a.e. $[\mu]$. Thus, $U^*_{(\cdot)} \in s^{\infty}(v^*)$ (i.e., implying that $(v^*, U^*_{(\cdot)}) \in Grs^{\infty}(\cdot)$.

(2) Next, for $s^{\infty}(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$, we will show that if the dominating probability measure, μ, is nonatomic, then for each v, $s^{\infty}(v)$ is contractible.

First, if the dominating probability measure, μ, is nonatomic, then as shown by Fryszkowski (1983), Liapunov's Theorem (1940) on the range of a vector measure guarantees the existence of a family of measurable sets, $\{E_t\}_{t \in [0,1]}$, such that

$$\left.\begin{array}{c} t' \leq t \Rightarrow E_{t'} \subseteq E_t,\ E_0 = \varnothing \text{ and } E_1 = \Omega, \text{ and} \\ \mu(E_t) = t\mu(\Omega) = t. \end{array}\right\} \tag{14}$$

Using the properties of this system of measurable sets and the decomposability of $s^{\infty}(v)$ for each $v \in \mathcal{L}^{\infty}_X$, we will show that for each v the function $h_v(\cdot, \cdot)$ given by

$$h_v(U, t) := U^1_{(\cdot)} I_{E_t}(\cdot) + U_{(\cdot)} I_{\Omega \setminus E_t}(\cdot) \in s^{\infty}(v) \text{ for all } (U, t) \in s^{\infty}(v) \times [0, 1] \tag{15}$$

is a homotopy (and in particular, a contraction of $s^{\infty}(v)$ to U^1). Here $v \in \mathcal{L}^{\infty}_X$ is fixed, $I_E(\cdot)$ is the indicator function of set E, and $U^1_{(\cdot)}$ is any fixed selection in $s^{\infty}(v)$.

It suffices to show that $h_v(\cdot,\cdot)$ is $\rho_{w^*\times|\cdot|}$-ρ_{w^*}-continuous. Let $\{(U^n_{(\cdot)},t^n)\}_n$ be such a sequence such that

$$U^n_{(\cdot)} \underset{w^*}{\longrightarrow} U^*_{(\cdot)} \text{ and } t^n \underset{R}{\longrightarrow} t^*.$$

We must show that

$$h_v(U^n_{(\cdot)},t^n) \underset{w^*}{\longrightarrow} h_v(U^*_{(\cdot)},t^*) \in s^\infty(v).$$

It suffices to show that for all $l \in \mathcal{L}^1_{R^m}$ with $\|l\|_1 \leq 1$,

$$H = \int_\Omega \left\langle \left(U^1_\omega I_{E_{t^n}}(\omega) - U^1_\omega I_{E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)$$

$$+ \int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega) \longrightarrow 0.$$

Rewriting, expression H, we have

$$H = \underbrace{\int_\Omega \left\langle \left(U^1_\omega I_{E_{t^n}}(\omega) - U^1_\omega I_{E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(a)}$$

$$+\underbrace{\int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^n}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(b)}$$

$$+\underbrace{\int_\Omega \left\langle \left(U^*_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega).}_{(c)}$$

Because $U^n \underset{w^*}{\longrightarrow} U^*$, we have

$$\int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^n}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)$$

$$+ \int_\Omega \left\langle \left(U^n_\omega I_{E_{t^n}}(\omega) - U^*_\omega I_{E_{t^n}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)$$

$$= \int_\Omega \left\langle \left(U^n_\omega - U^*_\omega\right), l(\omega)\right\rangle d\mu(\omega) \longrightarrow 0.$$

Thus, $(b) \longrightarrow 0$. Given that $X_d = [-M, M]$ for all d, we note that each of the expressions (a) and (c) is less than or equal to $2M\|l\|_1 \mu(E_{t^n} \bigtriangleup E_{t^*})$, and given that $\|l\|_1 \leq 1$, we have $(a) + (c) \leq 4M\mu(E_{t^n} \bigtriangleup E_{t^*})$. We have, then,

$$\int_\Omega \left\langle \left(U^1_\omega I_{E_{t^n}}(\omega) - U^1_\omega I_{E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)$$

$$+ \int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)$$

$$\leq 4M\mu(E_{t^n} \bigtriangleup E_{t^*}) + \int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^n}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega),$$

and as n goes to infinity

$$4M\mu(E_{t^n} \bigtriangleup E_{t^*})$$

$$+ \int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^n}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega) \longrightarrow 0.$$

Thus, the $\rho_{w^*\times|\cdot|}$-ρ_{w^*}-continuous function given in (15) for each $v \in \mathcal{L}^\infty_X$, together with the properties of the Liapunov system (14) specify a homotopy for the set of measurable selections, $s(v)$ - and thus for each v, $s(v)$ is contractible. Q.E.D.

Our proof that $s(v)$ is contractible for each v is a modified version of the proof given by Mariconda (1992) showing that if the underlying probability space is nonatomic then any decomposable subset of E-valued, Bochner integrable functions in $\mathcal{L}^1_E$ is contractible (where E is a Banach space). In Mariconda's result, the space of functions is equipped with the norm in $\mathcal{L}^1_E$, while here our space of functions (with each function taking values in $X \subset R^m$) is equipped with the metric, ρ_{w^*}, a metric compatible with the w^* topology.

3.3.1 Approximable Nash Payoff Selection Correspondences

The importance of the K-limit property derives from the fact that it guarantees w^*-w^*-contractibility and this in turn guarantees approximability, as our next result shows.

Theorem 3 *(If $s(\cdot) \in \mathcal{SC}[\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})]$ is a K correspondence and μ nonatomic, then $s(\cdot)$ is w^*-w^*-approximable)*
Let $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^\infty_X}$ be the parameterized one-shot game underlying a discounted stochastic game satisfying assumptions [DSG-1] and let

$$\mathcal{S}^\infty(\mathcal{P}_{(\cdot)}) : \mathcal{L}^\infty_X \longrightarrow P(\mathcal{L}^\infty_X),$$

be the Nash payoff selection correspondence. If $s^\infty(\cdot) \in \mathcal{SC}[\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})]$ is a K-correspondence, and if the dominating probability measure, μ, is nonatomic, then the Nash payoff selection sub-correspondence,

$$v \longrightarrow s^\infty(v),$$

is a w^*-w^*-approximable.

PROOF: By Corollary 5.6 in Gorniewicz, Granas, and Kryszewski (1991), because the sub-USCO, $s^\infty(\cdot)$, is defined on the ANR (absolute neighborhood retract) space of value functions $\mathcal{L}^\infty_X$ taking nonempty, compact, and contractible values in $\mathcal{L}^\infty_X$ (and hence ∞-proximally connected values - see Theorem 5.3 in Gorniewicz, Granas, and Kryszewski, 1991), the sub-USCO, $s^\infty(\cdot)$, is a J mapping. Therefore, by Theorem 5.12 in Gorniewicz, Granas, and Kryszewski (1991), $s^\infty(\cdot)$ is w^*-w^*-approximable. Q.E.D.

We can now state our main fixed point result.

Theorem 4 *(Fixed points for Nash payoff selection correspondences)*
Let $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^\infty_X}$ be the parameterized one-shot game underlying a discounted stochastic game satisfying assumptions [DSG-1] and having a Nash payoff selection correspondence,

$$\mathcal{S}^\infty(\mathcal{P}_{(\cdot)}) : \mathcal{L}^\infty_X \longrightarrow P(\mathcal{L}^\infty_X),$$

If $s^\infty(\cdot) \in \mathcal{SC}[\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})]$ is a K-correspondence, and if the dominating probability measure, μ, is nonatomic, then $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ has a fixed point (i.e., there exists $v^* \in \mathcal{L}^\infty_X$ such that $v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*})$).

PROOF: By Theorem 3 above, $s^\infty(\cdot)$ is w^*-w^*-approximable. Therefore, we have for each n, a w^*-w^*-continuous function,

$$g^n(\cdot) : \mathcal{L}^\infty_X \longrightarrow \mathcal{L}^\infty_X,$$

such that for each $(v^n, U^n) \in Grg^n \subset \mathcal{L}^\infty_X \times \mathcal{L}^\infty_X$ (i.e., for each $(v^n, U^n) \in \mathcal{L}^\infty_X \times \mathcal{L}^\infty_X$, with $U^n = g^n(v^n) \in \mathcal{L}^\infty_X$) there exists

$$(\overline{v}^n, \overline{U}^n) \in Grs(\cdot) \subset \mathcal{L}^\infty_X \times \mathcal{L}^\infty_X$$

(i.e., there exists $\overline{U}^n \in\ s^\infty(\overline{v}^n)$) such that

$$\rho_{w^*}(v^n, \overline{v}^n) + \rho_{w^*}(U^n_{(\cdot)}, \overline{U}^n_{(\cdot)}) < \tfrac{1}{n^2}. \tag{16}$$

Equivalently, for any positive integer, n,

$$Grg^n \subset B_{w^* \times w^*}(\tfrac{1}{n^2}, Grs^\infty(\cdot)).$$

Thus, the graph of the continuous function $g^n : \mathcal{L}^\infty_X \longrightarrow \mathcal{L}^\infty_X$ is contained in the $\rho_{w^* \times w^*}$-open ball of radius $\frac{1}{n^2}$ about the graph of $s^\infty(\cdot)$.

Because each of the functions, g^n, is w^*-w^*-continuous and defined on the w^*-compact and convex subset, $\mathcal{L}^\infty_X$, in $\mathcal{L}^\infty_{R^m}$, taking values in $\mathcal{L}^\infty_X$, it follows from the fixed point theorem of Schauder (see Aliprantis and Border, 2006), that each g^n has a fixed point, $v^n \in \mathcal{L}^\infty_X$ (i.e., for each n there exists some $v^n \in \mathcal{L}^\infty_X$ such that $v^n = g^n(v^n)$). Let $\{v^n\}_n$ be a fixed point sequence corresponding to the sequence of w^*-w^*-continuous approximating functions, $\{g^n(\cdot)\}_n$. Expression (16) can now be rewritten as follows: for each v^n in the fixed point sequence, there is a corresponding pair, $(\overline{v}^n, \overline{U}^n_{(\cdot)}) \in Grs^\infty(\cdot)$, such that

$$\rho_{w^*}(v^n, \overline{v}^n) + \rho_{w^*}(g^n(v^n), \overline{U}^n_{(\cdot)}) < \tfrac{1}{n^2},$$

and therefore such that

$$\underbrace{\rho_{w^*}(v^n, \overline{v}^n)}_{A} + \underbrace{\rho_{w^*}(v^n, \overline{U}^n_{(\cdot)})}_{B} < \tfrac{1}{n^2}. \tag{17}$$

By the w^*-compactness of $\mathcal{L}^\infty_X$, we can assume WLOG that the fixed point sequence, $\{v^n\}_n \subset \mathcal{L}^\infty_X$, w^*-converges to a limit $v^* \in \mathcal{L}^\infty_X$. Thus, by part A of (17), as $n \longrightarrow \infty$ we have

$$v^n \underset{w^*}{\longrightarrow} v^* \text{ and } \overline{v}^n \underset{w^*}{\longrightarrow} v^*,$$

and therefore by part B of (17), as $n \longrightarrow \infty$ we have

$$\overline{U}^n \underset{w^*}{\longrightarrow} v^*.$$

Because $Grs^\infty(\cdot)$ is $\rho_{w^* \times w^*}$-closed in $\mathcal{L}^\infty_X \times \mathcal{L}^\infty_X$,

$$\{(\overline{v}^n, \overline{U}^n)\}_n \subset Gr\mathcal{S}^\infty(\mathcal{P}_{(\cdot)}),$$

and $\overline{v}^n \underset{w^*}{\longrightarrow} v^*$ and $\overline{U}^n \underset{w^*}{\longrightarrow} v^*$ imply that

$$(v^*, v^*) \in Grs^\infty(\cdot).$$

Therefore,

$$v^* \in s^\infty(v^*) \subset \mathcal{S}^\infty(\mathcal{P}_{v^*}).$$

Q.E.D.

Given assumptions [DSG-1] it follows from Theorem 1 (Blackwell's Theorem) and Theorems 4 above that *all K-class discounted stochastic games have stationary Markov equilibria* (remember, as a consequence of Nowak, 2003, the K-limit property is sufficient to guarantee that $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ has fixed points).

4 Conditions Sufficient for a DSG to be K-Class

Thus far we have shown that if the graph of the Nash payoff selection correspondence, $Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, contains a K-closed subset, $s^{\infty} \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$ whose domain is all of $\mathcal{L}_X^{\infty}$ (i.e., $proj_{\mathcal{L}_X^{\infty}}(s^{\infty}) = \mathcal{L}_X^{\infty}$), then

$$v \longrightarrow s^{\infty}(v) := \left\{U_{(\cdot)} \in \mathcal{L}_X^{\infty} : (v, U_{(\cdot)}) \in s^{\infty}\right\}$$

is a w^*-w^*-sub-USCO taking contractible values belonging to $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$. Are there conditions sufficient to guarantee the existence of a K-closed set, s^{∞}, with

$$s^{\infty} \subseteq Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}?$$

We turn now to consider conditions sufficient to guarantee that the Nash payoff selection correspondence is a K-correspondence.

4.1 Noisy Discounted Stochastic Games

An interesting sub-class of discounted stochastic games is the class of noisy DSGs recently studied by Duggan (2012). In this subsection, we will show that all noisy discounted stochastic games are K-class. By specializing primitives and assumptions of our discounted stochastic game model, we can easily make our model a noisy stochastic game model. We need only modify assumptions (2) and (10) as follows:

In a noisy DSG (i.e., NDSG) the state space is given by $\Omega := T \times S$ with typical element $t := (t, s)$, where both T and S are complete separable metric spaces with metrics ρ_T and ρ_S, equipped with the Borel σ-fields B_T and B_S. By structuring the state space in this way, we can analyze situations where part of the riskiness is controllable (in a stochastic sense) and part of the riskiness is only indirectly controllable or not controllable at all. In particular, we can think of $t \in T$ as being the stochastically controllable regular state, and we can think of $s \in S$ as being the indirectly stochastically controllable (or uncontrollable) noisy state.

In an NDSG, the law of motion

$$(\underbrace{(t, s)}_{\omega}, a) \longrightarrow q(\cdot|\underbrace{(t, s)}_{\omega}, a)$$

is given by

$$q(d(t', s')|(t, s), a) := \varepsilon(ds'|t')\delta(dt'|(t, s), a),$$

or

$$q(d(t', s')|\omega, a) := \varepsilon(ds'|t')\delta(dt'|\omega, a),$$

where $\omega = (t, s)$ denotes the current state and $\omega' = (t', s')$ denotes the coming state - and depending on the regular state t' chosen by the probability measure, $\delta(dt'|\omega, a)$, in current state $\omega = (t, s)$ given action profile $a \in \Phi(\omega)$, the noisy state s' will be chosen according to the probability measure, $\varepsilon(ds'|t')$. Thus, while regular states are directly stochastically controllable via the stochastic kernel, $\delta(dt'|\omega, a)$, noisy states are only indirectly stochastically controllable via $\varepsilon(ds'|t')$. In this sense, we say that the discounted stochastic game is noisy.

To complete our formal description of the noisy discounted stochastic game model, assume that for all $t' \in T$, the probability measure, $\varepsilon(ds'|t')$, governing the choice of the coming noisy state s' is absolutely continuous with respect to a probability measure, $\lambda(ds')$, defined on the measurable space, (S, B_S), of noisy states.[6] Also, assume that for all

[6] Duggan assumes that the dominating probability measure, λ, is nonatomic - but we will show that this is not required for existence of stationary Markov equilibria.

$(\omega, a) \in \Omega \times \Phi(\omega)$, the probability measure, $\delta(dt'|\omega, a)$, governing the choice of the coming regular state t', given current state, $\omega := (t, s)$ and action profile $a \in \Phi(\omega)$, is absolutely continuous with respect to a probability measure, $\gamma(dt')$, defined on the measurable space, (T, B_T), of regular states. Thus, the noisy DSG has dominating probability measure given by the product measure,

$$\mu := \lambda \times \gamma.$$

By the Corollary in Rao and Rao (1972), if λ is nonatomic, then μ is nonatomic.[7]

Let

$$G_\gamma := \{g(dt|\omega, a) : (\omega, a) \in \Omega \times A\},$$

be the collection of probability density functions of $\delta(\cdot|\omega, a)$ with respect to γ such that for each state $\omega := (t, s)$, the function

$$(a_d, a_{-d}) \longrightarrow g(t'|(t, s), a_d, a_{-d})$$

is continuous in a and affine in a_d a.e. $[\gamma]$ in t'. Also, let

$$K_\lambda := \{r(ds'|t') : t' \in T\},$$

be the collection of probability density functions of $\varepsilon(\cdot|t')$ with respect to λ such that the function

$$t' \longrightarrow r(s'|t')$$

is measurable in t' a.e. $[\lambda]$ in s'.

Specializing (2) and (10) in our list of assumptions, [DSG-1], above, label the new list of assumptions [NSG-1].

Our first result establishes that all noisy DSGs have Nash payoff selection correspondences that are K-correspondences.

Theorem 5 *(For all noisy DSGs, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is a K-correspondence)*
Suppose assumptions [DSG-1] hold. If the DSG is noisy, then the Nash payoff selection correspondence,

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v)) := \mathcal{S}^\infty(U(\cdot, \mathcal{N}(\cdot, v), v)),$$

is a K correspondence.

PROOF: Let

$$\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^\infty(\mathcal{P}_{(\cdot)}),$$

be such that $\{(v^n, U^n_{(\cdot)})\}_n$ K-converges to K-limit $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^\infty_X \times \mathcal{L}^\infty_X$. We must show that $\widehat{U}_\omega \in Ls\{U^n_\omega\}$ a.e. $[\mu]$.

First, let $\{a^n(\cdot)\}_n$ be the sequence of measurable, A-valued functions such that for each n,

$$a^n(\omega) \in \mathcal{N}(\omega, v^n) \text{ for all } \omega$$

and

$$U^n_\omega = U(\omega, a^n(\omega), v^n) \text{ a.e. } [\mu],$$

for all n. Let N^∞ be the exceptional set. Thus, for all n,

$$U^n_\omega = U(\omega, a^n(\omega), v^n) \text{ for all } \omega \in \Omega \backslash N^\infty.$$

[7] $E \subset S$ is an atom of S relative to $\lambda(\cdot)$ if the following implication holds: if $\lambda(E) > 0$, then $H \subset E$ implies that $\lambda(H) = 0$ or $\lambda(E - H) = 0$. If S contains no atoms relative to $\lambda(\cdot)$, S is said to be atomless or nonatomic. Because S, is a complete, separable metric space $\lambda(\cdot)$ is atomless (or nonatomic) if and only if $\lambda(\{\omega\}) = 0$ for all $\omega \in S$ (see Hildenbrand, 1974, pp 44-45).

Next, recall

$$G_\gamma := \{g(dt'|(t,s),a) : ((t,s),a) \in (T \times S) \times A\ \}$$

be the collection of densities of $\delta(dt'|(t,s),a)$ with respect to probability measure $\gamma(\cdot)$ - under assumptions [DSG-1](10) we know that for each state $\omega := (t,s) \in T \times S$ the function

$$a := (a_d, a_{-d}) \longrightarrow g(t'|(t,s),a_d,a_{-d}) \text{ is continuous in } a \text{ and affine in } a_d$$

a.e.$[\gamma]$ in t'.

For the sequence of valuation functions , $\{v^n\}_n$, construct a new sequence, $\{V^n\}_n$, by letting

$$V^n(t') := \int_S v^n(t',s')r(s'|t') \in R^m,$$

and consider the sequence of functions, $\{F^n(\cdot,\cdot)\}_n$, given by

$$(\omega,t') \longrightarrow F^n(\omega,t') := ((1-\beta_d)r_d(\omega,a^n(\omega)) + \beta_d V_d^n(t')g(t'|\omega,a^n(\omega)))_{d\in D}\,.$$

For each ω, the sequence of R^m-valued integrable functions, $\{V_d^n(\cdot)g(\cdot|\omega,a^n(\omega))\}_n \subset \mathcal{L}^1_{R^m}$, is uniformly integrable and therefore $\mathcal{L}^1_{R^m}$-norm bounded. Moreover, for each ω, $a^{n_k}(\omega) \underset{\rho_A}{\longrightarrow} a^*(\omega)$ implies that for each player d

$$\textstyle\int_T V_d^{n_k}(t')g(t'|\omega,a^{n_k}(\omega))d\gamma(t') \longrightarrow \int_T \widehat{V}_d(t')g(t'|\omega,a^*(\omega))d\gamma(t').$$

Consider the sequence of 3-tuples,

$$\omega \longrightarrow \{r(\omega,a^n(\omega)), v^n(\omega), F^n(\omega,\cdot)\} \in X \times X \times \mathcal{L}^1_{R^m},$$

where for each n, $\omega \longrightarrow F^n(\omega,\cdot)$ is a mapping from Ω into $\mathcal{L}^1_{R^m}$. By Balder's (1990) extension of Komlos (1967), and by Komlos (1967), we can assume without loss of generality that there exists an exceptional set, $\widehat{N}$, with $\mu(\widehat{N}) = 0$, and three functions (Komlos limit functions), $(\widehat{r}(\cdot), \widehat{v}(\cdot), \widehat{F}(\cdot,\cdot))$, such that

$$\omega \longrightarrow (\widehat{r}(\omega), \widehat{v}(\omega), \widehat{F}(\omega,\cdot)) \in X \times X \times \mathcal{L}^1_{R^m}$$

such that for all $\omega \in \Omega\backslash\widehat{N} := (T \times S)\backslash\widehat{N}$,

$$\widehat{r}^n(t,s) := \tfrac{1}{n}\textstyle\sum_{k=1}^n r(t,s,a^k(t,s)) \longrightarrow \widehat{r}(t,s) \text{ by Komlos (1967),}$$

$$\widehat{v}^n(t,s) := \tfrac{1}{n}\textstyle\sum_{k=1}^n v^k(t,s) \longrightarrow \widehat{v}(t,s) \text{ by Komlos (1967),}$$

$$\tfrac{1}{n}\textstyle\sum_{k=1}^n F^k(t,s,\cdot) \underset{w}{\longrightarrow} \widehat{F}(t,s,\cdot) \text{ by Balder (1990).}$$

We have (see Ash, 1972)

$$\mu(\widehat{N}) = \int_S \lambda(\widehat{N}(t'))d\gamma(t') = 0,$$

where

$$\widehat{N}(t') := \left\{s' \in S : (t',s') \in \widehat{N}\right\},$$

implying that for some $\widehat{T}$ with $\gamma(\widehat{T}) = 0$, $\lambda(\widehat{N}(t')) = 0$ for all $t' \in T\backslash\widehat{T}$. Thus for each $t' \in T\backslash\widehat{T}$,

$$\widehat{v}^n(t',s') \longrightarrow \widehat{v}(t',s') \text{ for } s' \text{ a.e. } [\lambda].$$

Because for all t', $r(ds'|t')$ is absolutely continuous with respect to λ, we have for each $t' \in T \backslash \widehat{T}$,

$$\widehat{V}^n(t') := \frac{1}{n}\sum_{k=1}^{n} \int_S v^k(t', s')r(s'|t')$$

$$= \int_S [\frac{1}{n}\sum_{k=1}^{n} v^k(t', s')]r(s'|t')$$

$$= \int_S \widehat{v}^n(t', s')r(s'|t')$$

$$\longrightarrow \int_S \widehat{v}(t', s')r(s'|t')$$

$$:= \widehat{V}(t').$$

We have therefore, $V^n(\cdot) \underset{K}{\longrightarrow} \widehat{V}(\cdot)$ implying that $V^n(\cdot) \underset{w^*}{\longrightarrow} \widehat{V}(\cdot)$.

We have by Balder's extension of Komlos (see Theorem 4.1 in Balder, 1990), there exists a subsequence, $\{F^{n_k}(\cdot, \cdot)\}$, and a set, $\widehat{N}_n^*$, of μ-measure zero such that ,

$$\widehat{F}^n(\omega, \cdot) \underset{w}{\longrightarrow} \widehat{F}(\omega, \cdot) \in \mathcal{L}^1_{R^m}, \text{ for all } \omega \in \Omega \backslash \widehat{N}.$$

Therefore, for ω off of $\widehat{N}$,

$$\int_T \widehat{F}^n(\omega, t')d\gamma(t') \longrightarrow \int_T \widehat{F}(\omega, t')d\gamma(t') = \widehat{U}_\omega.$$

By Proposition 1 in Page (1991), for each ω off of $\widehat{N}$, there exists some $F^*(\omega, t') \in Ls\{F^n(\omega, t')\}$ a.e. $[\gamma]$ in t' such that

$$\int_T F^*(\omega, t')d\gamma(t') = \int_T \widehat{F}(\omega, t')d\gamma(t') = \widehat{U}_\omega.$$

Given the fact that that $V^n(\cdot) \underset{w^*}{\longrightarrow} \widehat{V}(\cdot)$, there exists some $a^*(\cdot) \in \Sigma(Ls\{a^n(\cdot)\}$, such that for each ω off of $\widehat{N}$

$$\int_T F^*(\omega, t')d\gamma(t') := \int_T \Big((1 - \beta_d r_d(\omega, a^*(\omega)) + \beta_d \widehat{V}_r(t')g(t'|\omega, a^*(\omega))\Big)_d d\gamma(t') \in Ls\{U^n_\omega\}.$$

Thus, we obtain the desired conclusion,

$$\widehat{U}_\omega \in Ls\{U^n_\omega\} \text{ for } \omega \in \Omega \backslash \widehat{N}.$$

Q.E.D.

Next will discuss an interesting measure theoretic condition introduced by Rokhlin (1949) and Dynkin and Evstigneev (1976) to ensure the existence of a convex set of conditional selections of a measurable, closed valued correspondence. We will call this condition the $\mathcal{G}$-nonatomic condition.

4.2 $\mathcal{G}$-Nonatomic Discounted Stochastic Games

Another interesting sub-class of DSGs recently studied by He and Sun (2013) is the class of $\mathcal{G}$-nonatomic DSGs. In this subsection we show that all $\mathcal{G}$-nonatomic DSGs are K-class. This sub-class, called by He and Sun (2013) games with a coaser transition kernels, derives its usefulness from the work of Dynkin and Evstigneev (1976) and Rokhlin (1949). The key ingredient making this so is the extension of Lyapunov's Theorem (1940) due to Dynkin and Evstigneev (1976). In what we do here, we go back to the definitions and results of Dynkin and Evstigneev (1976) - rather than He and Sun (2013). Recall that here we have assumed that the state space is a Polish space, Ω, equipped with the Borel σ-field, B_Ω, and a probability measure, μ, defined on B_Ω. Also, recall that when Ω is Polish, μ is nonatomic if and only if $\mu(\{\omega\}) = 0$ for all $\omega \in \Omega$ (see Hildenbrand, 1974). Suppose now that $\mathcal{G}$ is a sub-σ-field of B_Ω. Denote by $\mu^{\mathcal{G}}(\cdot)$ a regular $\mathcal{G}$-conditional probability given sub-σ-field $\mathcal{G}$. Following Dynkin and Evstigneev, $A \in B_\Omega$ is $\mathcal{G}$-atom if $\mu(A) > 0$ and for any $B \in B_\Omega$ such that $B \subset A$

$$\mu\left\{\omega \in \Omega : 0 < \mu^{\mathcal{G}}(B)(\omega) < \mu^{\mathcal{G}}(A)(\omega)\right\} = 0.$$

Let $\Gamma : \Omega \longrightarrow P_f(X)$ be an arbitrary measurable correspondence taking nonempty, closed values in X. We will denote by

$$\mathcal{S}^\infty_{\mathcal{G}}(\Gamma) := \{E(U|\mathcal{G}) \in \mathcal{L}^\infty_X(\mathcal{G}) : U \in \mathcal{S}^\infty(\Gamma)\}$$

the collection of all μ-equivalence classes of regular $\mathcal{G}$-conditional expectations of μ-essentially bounded a.e. measurable selections of Γ. The following extension of Lyapunov's Theorem is due to Dynkin and Evstigneev (1976).

Theorem 6 *(An extension of Lyapunov's Convexity Theorem)*
Let $\Gamma : \Omega \longrightarrow P_f(X)$ be a measurable correspondence taking nonempty, closed values in X. If for some sub-σ-field, $\mathcal{G}$, of B_Ω, B_Ω contains no $\mathcal{G}$-atoms, then

$$\mathcal{S}^\infty_{\mathcal{G}}(\Gamma) = \mathcal{S}^\infty_{\mathcal{G}}(co\Gamma),$$

where co denotes the convex hull.

He and Sun (2013) give a slightly different definition of $\mathcal{G}$-atoms - one implied by Dynkin and Evstigneev's definition - and they show that if the state space underlying the game is nonatomic and has no $\mathcal{G}$-atoms, then the the discounted stochastic game has a stationary Markov equilibrium. Our next result, a Corollary of Theorem 6, shows that our condition, the K-limit property, is implied by the absence of $\mathcal{G}$-atoms.

Corollary to Theorem 6 *(All $\mathcal{G}$-nonatomic DSGs are K-class nonatomic DSGs)*
Let $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^\infty_X}$ be the parameterized one-shot game underlying a discounted stochastic game, DSG, satisfying assumptions [DSG-1] and having a Nash payoff selection correspondence,

$$\mathcal{S}^\infty(\mathcal{P}_{(\cdot)}) : \mathcal{L}^\infty_X \longrightarrow P(\mathcal{L}^\infty_X).$$

If the underlying probability space, (Ω, B_Ω, μ), is such that for some sub-σ-field, $\mathcal{G}$, of B_Ω, B_Ω contains no $\mathcal{G}$-atoms, then $\mathcal{S}^\infty_{\mathcal{G}}(\mathcal{P}_{(\cdot)})$ has the K-limit property, and in fact, is a convex-valued, w^*-w^*-sub-USCO of $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$.

PROOF: Let $\{(v^n, U^n_{(\cdot)})\}_n$ be any sequence contained in $Gr(\mathcal{S}^\infty(\mathcal{P}_{(\cdot)}))$ such that $v^n \underset{w^*}{\longrightarrow} v^* \in \mathcal{L}^\infty_X$. We have for each n, $U^n(\omega) \in \mathcal{P}(\omega, v^n)$ a.e. $[\mu]$. By the K compactness of $\mathcal{L}^\infty_X$, we can assume WLOG that the sequence, $\{(v^n, U^n_{(\cdot)})\}_n$, K converges with K limit $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^\infty_X \times \mathcal{L}^\infty_X$. We have

$$\widehat{U}^n(\omega) := \frac{1}{n}\sum_{k=1}^{n} U^k(\omega) \longrightarrow \widehat{U}(\omega) \text{ a.e. } [\mu],$$

and by the properties of conditional expectations (see Ash, 1972),

$$E(\widehat{U}^n|\mathcal{G})(\omega) := \frac{1}{n}\sum_{k=1}^{n} E(U^k|\mathcal{G})(\omega) \longrightarrow E(\widehat{U}|\mathcal{G})(\omega) \text{ a.e. } [\mu].$$

By Proposition 1 in Page (1991),

$$E(\widehat{U}|\mathcal{G})(\omega) \in coLs\{E(U^n|\mathcal{G})(\omega)\} \text{ a.e. } [\mu].$$

By Dynkin and Evstigneev (1976),

$$coLs\{E(U^n|\mathcal{G})(\omega)\} = Ls\{E(U^n|\mathcal{G})(\omega)\} \text{ a.e. } [\mu].$$

and by the properties of conditional expectations, $coLs\{E(U^n|\mathcal{G})(\cdot)\} \subset \mathcal{S}^{\infty}_{\mathcal{G}}(\mathcal{P}_{v^*})$. Thus,

$$E(\widehat{U}|\mathcal{G})(\omega) \in Ls\{E(U^n|\mathcal{G})(\omega)\} \text{ a.e. } [\mu],$$

i.e., $\mathcal{S}^{\infty}_{\mathcal{G}}(\mathcal{P}_{(\cdot)})$ has the K-limit property. In fact, $\mathcal{S}^{\infty}_{\mathcal{G}}(\mathcal{P}_{(\cdot)})$ is a convex-valued, and by Theorem A2.3 is a w^*-w^*-sub-USCO of $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$. Q.E.D.

5 Appendix 1: Mathematical Preliminaries

5.1 Hyperspaces

Let X be a nonempty, closed subset of R^m and let $P_f(X)$ denote the hyperspace of all nonempty, closed (and hence compact) subsets of X. Equip $P_f(X)$ with the Hausdorff metric h_X generated by the sum metric, ρ_X, on X.[8] Because (X, ρ_X) is a compact metric space, so too is the hyperspace, $(P_f(X), h_X)$.

Also, let $(P_{w^*f}(\mathcal{L}^{\infty}_X), h_{w^*})$ denote the hyperspace of nonempty, ρ_{w^*}-closed subsets of $\mathcal{L}^{\infty}_X$ equipped with the Hausdorff metric generated by the underlying metric ρ_{w^*}. Because $(\mathcal{L}^{\infty}_X, \rho_{w^*})$ is a compact metric space, the hyperspace, $(P_{w^*f}(\mathcal{L}^{\infty}_X), h_{w^*})$, is a compact metric space.

Finally, let $P_f(\mathcal{L}^{\infty}_X \times X)$ denote the collection of all nonempty, $\rho_{w^*\times X}$-closed subsets of $\mathcal{L}^{\infty}_X \times X$, where $\rho_{w^*\times X}$ denotes the sum metric, $\rho_{w^*} + \rho_X$, on $\mathcal{L}^{\infty}_X \times X$.[9] Equip $P_f(\mathcal{L}^{\infty}_X \times X)$ with Hausdorff metric, $h_{w^*\times X}$, generated by the metric, $\rho_{w^*\times X}$, on $\mathcal{L}^{\infty}_X \times X$. Again, because $(\mathcal{L}^{\infty}_X \times X, \rho_{w^*\times X})$ is a compact metric space, so too is $(P_f(\mathcal{L}^{\infty}_X \times X), h_{w^*\times X})$.

[8] The Hausdorff metric, h_X, on $P_f(X)$ is defined as follows: for all F and F' in $P_f(X)$, the Hausdorff distance between F and F' is given by

$$h_X(F, F') := \max\{e_X(F, F'), e_X(F', F)\}$$

where

the excess of F over F' is given by
$e_X(F, F') := \sup_{x \in F} dist_{\rho_X}(x, F')$,
and
the excess of F' over F is given by
$e_X(F', F) := \sup_{x' \in F'} dist_{\rho_X}(x', F)$,

and where

the distance from x to F' is given by
$dist_{\rho_X}(x, F') := \inf_{x' \in F'} \rho_X(x, x')$,
while the distance from x' to F is given by
$dist_{\rho_X}(x', F) := \inf_{x \in F} \rho_{w^*\times X}(x', x)$.

[9] Thus, for (v, u) and (v', u') in $\mathcal{L}^{\infty}_X \times X$,

$$\rho_{w^*\times X}((v, u), (v', u')) := \rho_{w^*}(v, v') + \rho_X(u, u').$$

5.2 Upper Caratheodory Correspondences

The correspondence,

$$\Gamma(\cdot,\cdot):\Omega\times\mathcal{L}_X^{\infty}\longrightarrow P_f(X),$$

is upper Caratheodory if $\Gamma(\cdot,\cdot)$ is product measurable on $\Omega\times\mathcal{L}_X^{\infty}$ and w^*-X-upper semicontinuous on $\mathcal{L}_X^{\infty}$, and for all

$$(\omega,v):=(\omega,(v_1,\ldots,v_m))\in\Omega\times\mathcal{L}_X^{\infty},$$

$\Gamma(\omega,v)\in P_f(X)$.

For the correspondence, $\Gamma(\cdot,\cdot)$, and subset F of X, let

$$\Gamma^-(F):=\{(\omega,v)\in\Omega\times\mathcal{L}_X^{\infty}:\Gamma(\omega,v)\cap F\neq\varnothing\}.$$

The correspondence, $\Gamma(\cdot,\cdot)$, is product measurable if for each *open* subset, E, of X, $\Gamma^-(E)\in B_\Omega\times B_X$, where B_X is the Borel product σ-field, $B_X:=B_{X_1}\times\cdots\times B_{X_m}$ in X. For fixed ω, the correspondence, $\Gamma(\omega,\cdot):=\Gamma_\omega(\cdot)$, is w^*-X-upper semicontinuous if $\Gamma^-(\omega,F):=\Gamma_\omega^-(F)$ is w^*-closed for each closed subset, F, of X.[10] The correspondence, $\Gamma(\cdot,\cdot)$, is *upper Caratheodory* if it is product measurable in ω and v and w^*-X-upper semicontinuous in v for each ω (when no confusion is possible we will write upper semicontinuous rather than w^*-X-upper semicontinuous).

Let B_{w^*} be the Borel σ-field in $\mathcal{L}_X^{\infty}$ generated by the weak star open (i.e., the ρ_{w^*}-open) subsets of $\mathcal{L}_X^{\infty}$. A function

$$g:\Omega\longrightarrow\mathcal{L}_X^{\infty}$$

is (B_Ω,B_{w^*})-measurable (or just measurable) if for any Borel measurable subset $\mathcal{E}\in B_{w^*}$ of $\mathcal{L}_X^{\infty}$,

$$\{\omega\in\Omega:g(\omega)\in\mathcal{E}\}\in B_\Omega.$$

Similarly, a function

$$f:\Omega\longrightarrow X$$

is (B_Ω,B_X)-measurable (or just measurable) if for any Borel measurable subset E of X,

$$\{\omega\in\Omega:f(\omega)\in E\}\in B_\Omega.$$

5.3 USCOs

Consider the correspondence,

$$\Gamma(\cdot):\mathcal{L}_X^{\infty}\longrightarrow P_f(X).$$

We say that a correspondence (or a set-valued mapping) $\Gamma(\cdot)$ is an USCO if $\Gamma(\cdot)$ is upper semicontinuous (usc) with nonempty, compact values. Here, because X is compact, $\Gamma(\cdot)$ is an USCO if $\Gamma(\cdot)$ is upper semicontinuous (usc) with $\Gamma(v)\in P_f(X)$ for all $v\in L_X^{\infty}$. We will denote by

$$\mathcal{U}_{w^*\text{-}X}:=\mathcal{U}(\mathcal{L}_X^{\infty},P_f(X))$$

the set of all USCOs defined on $\mathcal{L}_X^{\infty}$ with values contained in $P_f(X)$ (see Anguelov and Kalenda, 2009, Crannell, Franz, and LeMasurier, 2005 and Hola and Holy, 2009). Note

[10] Equivalently, $\Gamma(\omega,\cdot)$ is w^*-X-upper semicontinuous (usc) if given any open subset G of X, the set

$$\Gamma_\omega^+(G):=\{v\in\mathcal{L}_X^{\infty}:\Gamma(\omega,v)\subset G\}$$

is w^*-open in $\mathcal{L}_X^{\infty}$.

also that because X is compact, $\Gamma(\cdot)$ is a USCO if and only if $Gr\Gamma(\cdot)$ is $w^* \times X$-closed (i.e., $Gr\Gamma(\cdot) \in P_f(\mathcal{L}_X^\infty \times X)$).[11] Note that if $(\omega, v) \longrightarrow \Gamma(\omega, v)$ is an upper Caratheodory mapping, then for each ω, the mapping,

$$v \longrightarrow \Gamma_\omega(v) := \Gamma(\omega, v),$$

is an USCO.

Finally, we will denote by

$$\mathcal{U}_{w^*\text{-}w^*} := \mathcal{U}(\mathcal{L}_X^\infty, P_{w^*f}(\mathcal{L}_X^\infty))$$

the collection of all USCOs,

$$\Gamma(\cdot) : \mathcal{L}_X^\infty \longrightarrow P_{w^*f}(\mathcal{L}_X^\infty),$$

defined on $\mathcal{L}_X^\infty$ taking nonempty, w^*-closed (and hence w^*-compact) values in $\mathcal{L}_X^\infty$.

Given an USCO, $\Gamma(\cdot)$, we say that $\gamma(\cdot)$ is a sub-USCO belonging to $\Gamma(\cdot)$ if $\gamma(\cdot)$ is an USCO and the graph of $\gamma(\cdot)$, denoted by $Gr\gamma(\cdot)$ is contained in the graph, $Gr\Gamma(\cdot)$, of the USCO $\Gamma(\cdot)$. Note that $\Gamma(\cdot)$ is a sub-USCO of itself.

5.4 w^*-w^*-USCOs

We say that a set-valued mapping (or a correspondence),

$$\Gamma(\cdot) : \mathcal{L}_X^\infty \longrightarrow P_{w^*f}(\mathcal{L}_X^\infty),$$

is w^*-w^*-upper semicontinuous if

$$\Gamma^-(F) := \{v \in \mathcal{L}_X^\infty : \Gamma(v) \cap F \neq \varnothing\}$$

is w^*-closed for each w^*-closed subset, F, of $\mathcal{L}_X^\infty$.[12]

We say that $\Gamma(\cdot)$ is a w^*-w^*-USCO if $\Gamma(\cdot)$ is w^*-w^*-upper semicontinuous (w^*-w^*-usc) with nonempty, w^*-compact values. Here, because $\mathcal{L}_X^\infty$ is w^*-compact, $\Gamma(\cdot)$ is an USCO if $\Gamma(\cdot)$ is w^*-w^*-upper semicontinuous with

$$\Gamma(v) \in P_{w^*f}(\mathcal{L}_X^\infty) \text{ for all } v \in \mathcal{L}_X^\infty.$$

We will denote by

$$\mathcal{U}_{w^*\text{-}w^*} := \mathcal{U}(\mathcal{L}_X^\infty, P_{w^*f}(\mathcal{L}_X^\infty))$$

the set of all w^*-w^*-USCOs defined on $\mathcal{L}_X^\infty$ with values contained in $P_{w^*f}(\mathcal{L}_X^\infty)$. Note that because $\mathcal{L}_X^\infty$ is w^*-compact, $\Gamma(\cdot)$ is a w^*-w^*-USCO if and only if $Gr\Gamma(\cdot)$ is $w^* \times w^*$-closed.[13]

[11] $Gr\Gamma(\omega, \cdot)$ is closed if for any sequence $\{(v^n, u^n)\}_n$ in $Gr\Gamma(\omega, \cdot)$, $v^n \underset{\rho_{w^*}}{\rightarrow} \overline{v}$ and $u^n \underset{\rho_X}{\rightarrow} \overline{u}$ imply that $(\overline{v}, \overline{u}) \in Gr\Gamma(\omega, \cdot)$.

[12] Equivalently, $\Gamma(\cdot)$ is w^*-w^*-upper semicontinuous (usc) if given any w^*-open subset G of $\mathcal{L}_X^\infty$, the set

$$\Gamma^+(G) := \{v \in \mathcal{L}_X^\infty : \Gamma(\omega, v) \subset G\}$$

is w^*-open in $\mathcal{L}_X^\infty$.

[13] $Gr\Gamma(\cdot)$ is $w^* \times w^*$-closed if for any sequence $\{(v^n, u^n)\}_n$ in $Gr\Gamma(\cdot)$, $v^n \underset{\rho_{w^*}}{\rightarrow} \overline{v}$ and $u^n \underset{\rho_{w^*}}{\rightarrow} \overline{u}$ imply that $(\overline{v}, \overline{u}) \in Gr\Gamma(\cdot)$.

5.5 Continuous Functions

A function $U : L^\infty_X \longrightarrow X$ is said to be w^*-X-continuous at $\overline{v} \in L^\infty_X$ if for every ρ_X-open subset G of X such that $U(\overline{v}) \in G$ there is a w^*-open set $N_{\overline{v}}$ containing $\overline{v}$ such that $U(N_{\overline{v}}) \subset G$. The function U is w^*-X-continuous if it is w^*-X-continuous at every $v \in L^\infty_X$.

A function $g : L^\infty_X \longrightarrow L^\infty_X$ is said to be w^*-$\|\cdot\|_1$-continuous at $\overline{v} \in L^\infty_X$ if for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for all $v \in B_{\rho_{w^*}}(\delta_\varepsilon, \overline{v}) \cap L^\infty_X$, $\|g(v) - g(\overline{v})\|_1 < \varepsilon$. The function g is w^*-$\|\cdot\|_1$-continuous if it is w^*-$\|\cdot\|_1$-continuous at every $v \in L^\infty_X$.

Finally, a function $g : L^\infty_X \longrightarrow L^\infty_X$ is said to be w^*-w^*-continuous at $\overline{v} \in L^\infty_X$ if for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for all $v \in B_{\rho_{w^*}}(\delta_\varepsilon, \overline{v}) \cap L^\infty_X$, $\rho_{w^*}(g(v), g(\overline{v})) < \varepsilon$. The function g is w^*-w^*-continuous if it is w^*-w^*-continuous at every $v \in L^\infty_X$.

We will denote by $\mathcal{C}_{w^*\text{-}X} := \mathcal{C}(\mathcal{L}^\infty_X, X)$ the collection of all w^*-X-continuous functions defined on $\mathcal{L}^\infty_X$ with values in X, by $\mathcal{C}_{w^*\text{-}\|\cdot\|_1} := \mathcal{C}(\mathcal{L}^\infty_X, \mathcal{L}^\infty_X)$ the collection of all w^*-$\|\cdot\|_1$-continuous functions defined on $\mathcal{L}^\infty_X$ with values in $\mathcal{L}^\infty_X$, and by $\mathcal{C}_{w^*\text{-}w^*} := \mathcal{C}(\mathcal{L}^\infty_X, \mathcal{L}^\infty_X)$ the collection of all w^*-w^*-continuous functions defined on $\mathcal{L}^\infty_X$ with values in $\mathcal{L}^\infty_X$.

Continuity Theorem A7.1.1 *(Continuity of Players' Payoff Profile Function)*
Let DSG be a discounted stochastic game satisfying assumptions [DSG-1] with players' payoff profile function,

$$(v, a) \longrightarrow U(\omega, a, v) := (U_1(\omega, a, v_1), \ldots, U_m(\omega, a, v_m)).$$

If $\{(v^n, a^n)\}_n$ is a sequence in $\mathcal{L}^\infty_X \times A$ such that $v^n \underset{w^*}{\longrightarrow} v^*$ and $a^n \underset{\rho_A}{\longrightarrow} a^*$, then in each state $\omega \in \Omega$,

$$U(\omega, a^n, v^n) \underset{R^m}{\longrightarrow} U(\omega, a^*, v^*).$$

Proof. Let $\{(v^n, a^n)\}_n$ be a sequence such that $v^n \underset{w^*}{\longrightarrow} v^*$ and $a^n \underset{\rho_A}{\longrightarrow} a^*$. Let ω be given and fixed, and observe that for each players d:

$$|U_d(\omega, a^n, v^n_d) - U_d(\omega, a^*, v^*_d)|_R$$

$$\leq \underbrace{|U_d(\omega, a^n, v^n_d) - U_d(\omega, a^*, v^n_d)|_R}_{A^n} + \underbrace{|U_d(\omega, a^*, v^n_d) - U_d(\omega, a^*, v^*_d)|_R}_{B^n}.$$

We will carry out our proof for one player d, keeping in mind that the argument holds for all players simultaneously. Consider B^n first. We have

$$B^n = \beta_d \left| \int_\Omega v^n_d(\omega') q(\omega' | \omega, a^*) - \int_\Omega v^*_d(\omega') q(\omega' | \omega, a^*) \right|_R.$$

Let $h(\cdot|\omega, a^*)$ be a density of $q(\cdot|\omega, a^*)$ with respect to μ. Given that $v^n_d \underset{w^*_d}{\longrightarrow} v^*_d$, we have,

$$\textstyle\int_\Omega v^n_d(\omega') q(\omega'|\omega, a^*) = \int_\Omega v^n_d(\omega') h(\omega'|\omega, a^*) d\mu(\omega')$$

$$\textstyle\to \int_\Omega v^*_d(\omega') h(\omega'|\omega, a^*) d\mu(\omega') = \int_\Omega v^*_d(\omega') q(\omega'|\omega, a^*).$$

Thus, $B^n \overset{n}{\to} 0$. Next, consider A^n. We have

$$A^n \leq (1 - \beta_d) \underbrace{|r_d(\omega, a^n) - r_d(\omega, a^*)|_R}_{A^n_1}$$

$$+ \underbrace{\beta_d \left| \int_\Omega v^n_d(\omega') q(\omega'|\omega, a^n) - \int_\Omega v^n_d(\omega') q(\omega'|\omega, a^*) \right|_R}_{A^n_2}.$$

Continuity of $r_d(\omega,\cdot)$ and $a^n \underset{\rho_A}{\longrightarrow} a^*$ imply that $A_1^n \overset{n}{\longrightarrow} 0$. To see that $A_2^n \overset{n}{\longrightarrow} 0$, observe that

$$\begin{aligned}&\left|\int_\Omega v_d^n(\omega')q(\omega'|\omega,a^n) - \int_\Omega v_d^n(\omega')q(\omega'|\omega,a^*)\right|\\ &\leq M\left\|q(\cdot|\omega,a^n) - q(\cdot|\omega,a^*)\right\|_\infty \overset{n}{\to} 0.\end{aligned}$$

Q.E.D.

6 Appendix 2: K-Convergence

6.1 The Relevant Function Spaces

Let (Ω, B_Ω, μ) be a probability space where Ω is Polish (complete, separable, metric) and μ is a probability measure defined on the Borel σ-field B_Ω.

We have

$$\mathcal{L}_X^\infty \subset \mathcal{L}_X^1$$
and
$$\mathcal{L}_X^\infty \subset \mathcal{L}_{R^m}^\infty \subset \mathcal{L}_{R^m}^1.$$

We note that if a sequence, $\{v^n\}_n$, in $\mathcal{L}_X^\infty$ converges in $\mathcal{L}^1$-norm to $\overline{v}$, then $\overline{v} \in \mathcal{L}_X^\infty$ - thus, $\mathcal{L}_X^\infty$ is $\mathcal{L}^1$-norm closed in $\mathcal{L}_{R^m}^\infty$ (see Pales and Zeidan 1999). Recall that a sequence $\{v^n\}_n$ in $\mathcal{L}_{R^m}^\infty$ w^*-converges (or converges weak star) to $v^\infty \in \mathcal{L}_{R^m}^\infty$ if for each $l^1 \in \mathcal{L}_{R^m}^1$,

$$\left\langle v^n, l^1\right\rangle := \int_\Omega \left\langle v^n(\omega), l^1(\omega)\right\rangle d\mu(\omega) \underset{R^m}{\longrightarrow} \int_\Omega \left\langle v^\infty(\omega), l^1(\omega)\right\rangle d\mu(\omega) := \left\langle v^\infty, l^1\right\rangle.$$

The space of value function profiles, $(\mathcal{L}_X^\infty, \rho_{w^*})$, is a compact, convex, metric space. We will denote by $\underset{w^*}{\longrightarrow}$ sequential convergence in $\mathcal{L}_X^\infty$ with respect to the metric ρ_{w^*} - and therefore, convergence in $\mathcal{L}_X^\infty$ with respect to the w^*-topology.

6.2 K-Convergence

In this subsection we state the fundamental definitions and results we will need about K-convergence.

6.2.1 K-Convergence in $\mathcal{L}_{R^m}^1$

Consider a sequence $\{U^n(\cdot)\}_n \subset L_{R^m}^1$ with corresponding sequence of arithmetic mean functions, $\left\{\frac{1}{n}\sum_{k=1}^n U^k(\cdot)\right\}_n$, and for any subsequence, $\{U^{n_k}(\cdot)\}_k$, of $\{U^n(\cdot)\}_n$, let the corresponding subsequence of arithmetic mean functions be given by

$$\left\{\frac{1}{k}\sum_{i=1}^k U^{n_i}(\cdot)\right\}_n.$$

Finally, for each n, let $\widehat{U}^n(\cdot) := \frac{1}{n}\sum_{k=1}^n U^k(\cdot)$ and for each k, let $\widehat{U}^{n_k}(\cdot) := \frac{1}{k}\sum_{i=1}^k U^{n_i}(\cdot)$.

Definition A7.2.1 (*K-Sequences, K-Convergence, and K-Limits*)
We say that a sequence $\{U^n\}_n \subset L_{R^m}^1$ is K-convergent (or is a K-sequence) if there exists a K-limit function $\widehat{U} \in L_{R^m}^1$ such that,
(a) the corresponding sequence of arithmetic mean functions, $\{\widehat{U}^n(\cdot)\}_n$, converges pointwise a.e. $[\mu]$ to $\widehat{U}(\cdot)$, that is,

$$\widehat{U}^n(\omega) \longrightarrow \widehat{U}(\omega) \text{ a.e. } [\mu],$$

and
(b) for any subsequence, $\{U^{n_k}(\cdot)\}_k$, of $\{U^n(\cdot)\}_n$, the corresponding subsequence of arithmetic mean functions, $\{\widehat{U}^{n_k}(\cdot)\}_n$, converges pointwise a.e. $[\mu]$ to $\widehat{U}(\cdot)$ as well, that is,

$$\widehat{U}^{n_k}(\omega) \longrightarrow \widehat{U}(\omega) \text{ a.e. } [\mu].$$

We will often refer to set of μ-measure zero where pointwise arithmetic mean convergence fails for a particular subsequence as the subsequence's *exceptional set.*

A sequence, $\{U^n(\cdot)\}_n$, of functions in $\mathcal{L}^1_{R^m}$ is norm-bounded provided

$$\sup_n \|U^n\|_1 := \sup_n \sum_{d=1}^m \|U^n_d\|_1 < \infty.$$

For the convenience of the reader we state the Theorem of Komlos (1967) as well as a variation on Artstein's Proposition C (1978) due to Page (1991).

Theorem A7.2.1 *(Komlos Theorem, 1967):*
If $\{U^n(\cdot)\}_n \subset \mathcal{L}^1_{R^m}$ is $\|\cdot\|_1$-bounded, then $\{U^n(\cdot)\}_n$ has a K-convergent subsequence.
Theorem A7.2.2 *(Page's Theorem, 1991):*
If the sequence $\{v^n(\cdot)\}_n \subset \mathcal{L}^1_{R^m}$ is $\|\cdot\|_1$-bounded and K-converges to some integrable R^m-valued function, $\widehat{v}(\cdot)$, then

$$\widehat{v}(\omega) \in coLs\{v^n(\omega)\} \text{ a.e. } [\mu]$$

and there exists an integrable R^m-valued function, $v^*(\cdot)$, such that $v^*(\omega) \in Ls\{v^n(\omega)\}$ a.e. $[\mu]$ and

$$\int_\Omega v^*(\omega)d\mu(\omega) = \int_\Omega \widehat{v}(\omega)d\mu(\omega).$$

We say that a set of functions, $\mathcal{H} \subset \mathcal{L}^1_{R^m}$, is K-compact if every sequence, $\{U^n\}_n \subset \mathcal{H}$, has a K-convergent subsequence with K-limit contained in $\mathcal{H}$. By Komlos' Theorem any $\|\cdot\|_1$-bounded subset $\mathcal{H}$ of $\mathcal{L}^1_{R^m}$ is relatively K-compact (i.e., has a K-converging subsequence with K-limit contained in $\mathcal{L}^1_{R^m}$).

6.2.2 K-convergence and w^*-Convergence in $\mathcal{L}^\infty_{R^m}$

Our next results concern the relationships between K-convergence and weak star (w^*-convergence) in $\mathcal{L}^\infty_{R^m}$.

Theorem A7.2.3 *(K-Convergence and w^*-Convergence):*
Suppose the primitives satisfy assumptions [DSG-1]. Let $\{v^n\}_n$ be any sequence in $\mathcal{L}^\infty_{R^m}$. Then the following statements are true:
(1) If $\{v^n\}_n$ K-converges to $\widehat{v} \in L^\infty_{R^m}$, then $\{v^n\}_n \subset \mathcal{L}^\infty_{R^m}$ w^*-converges to $\widehat{v} \in \mathcal{L}^\infty_{R^m}$.
(2) If $\{v^n\}_n \subset \mathcal{L}^\infty_{R^m}$ w^*-converges to $v^* \in \mathcal{L}^\infty_{R^m}$, then each K-convergent subsequence of $\{v^n\}_n$ has a K-limit, $\widehat{v} \in L^\infty_{R^m}$, such that $\widehat{v} = v^*$ a.e. $[\mu]$.

Before proceeding to the proof, some comments on notation: In the statement of the Theorem above, we write $\{v^n\}_n \subset \mathcal{L}^\infty_{R^m}$, to indicate that rather than viewing the sequence $\{v^n\}_n$ as a sequence of specific functions - which we will denote by $\{v^n\}_n \subset L^\infty_{R^m}$ - we are instead viewing the sequence as a sequence of μ-equivalence classes in $\mathcal{L}^\infty_{R^m}$ indexed by the specific functions, v^n. Thus, we write $\widehat{v} \in \mathcal{L}^\infty_{R^m}$ to denote the μ-equivalence class in $\mathcal{L}^\infty_{R^m}$ determined by the specific function, $\widehat{v}$.

PROOF: We will prove part (2) first. Assume that $\{v^n\}_n \subset \mathcal{L}^\infty_{R^m}$ w^*-converges to $v^\infty \in \mathcal{L}^\infty_{R^m}$, and that the subsequence, $\{v^{n_k}\}_k$, K-converges to $\widehat{v} \in L^\infty_{R^m}$. For each N and each $l \in L^1_{R^m}$ we have

$$\frac{1}{k}\sum_{i=1}^{k} v^{n_i}(\omega)l(\omega) \longrightarrow \widehat{v}(\omega)l(\omega) \text{ a.e. } [\mu]$$

and by the Dominated Convergence Theorem also in $\mathcal{L}^1_{R^m}$-norm. Thus, for each $l \in L^1_{R^m}$,

$$\int_\Omega \widehat{v}(\omega)l(\omega)d\mu(\omega)$$

$$:= \lim_{k\longrightarrow\infty} \tfrac{1}{k}\sum_{i=1}^{k} \int_\Omega v^{n_i}(\omega)l(\omega)d\mu(\omega)$$

$$= \int_\Omega v^\infty(\omega)l(\omega)d\mu(\omega),$$

and hence $\widehat{v}(\omega) = v^\infty(\omega)$ a.e. $[\mu]$.

Now we will prove part (1). Assume that $\{v^n\}_n \subset L^\infty_{R^m}$ K-converges to some $\widehat{v} \in L^\infty_{R^m}$. In order to show that $\{v^n\}_n$ w^*-converges to $\widehat{v}$, by w^*-compactness and metrizability, it suffices to show that the μ-equivalence class in $\mathcal{L}^\infty_{R^m}$ containing $\widehat{v}$ is the only limit point of the sequence of μ-equivalence classes, $\{v^n\}_n \subset \mathcal{L}^\infty_{R^m}$. Let v^∞ be any w^*-limit point of the sequence $\{v^n\}_n$ and let $\{v^{n_k}\}_k$ be a subsequence w^*-converging to v^∞. By K-convergence we know that this subsequence also K-converges to $\widehat{v}$ and hence by part (2) we know that $v^\infty = \widehat{v}$ a.e. $[\mu]$. Q.E.D.

7 Appendix 3: Metric Topology

7.1 Basics

Throughout assume that (Z, ρ_z) and (X, ρ_X) are compact metric spaces.[14] Because the space Z is compact, for any collection $\{G_\alpha\}_\alpha$ of open sets in Z where $Z = \cup_\alpha G_\alpha$ and α ranges over an arbitrary set A, there exists a finite subcollection, $\alpha_1, \ldots, \alpha_n$ such that $Z = G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$ (i.e., the Borel-Lebesgue condition - every open cover of Z contains a finite subcover). The Borel-Lebesgue condition is equivalent to the Riesz condition: if $\{F_\alpha\}_\alpha$ is a collection of closed sets in Z such that $\cap_\alpha F_\alpha = \emptyset$, then there is a finite subcollection, $\alpha_1, \ldots, \alpha_n$ such that $F_{\alpha_1} \cap \cdots \cap F_{\alpha_n} = \emptyset$ (see Kuratowski, 1972).

Let $\mathcal{C}(Z, X)$ denote the collection of continuous functions defined on Z taking values in X. If $f \in \mathcal{C}(Z, X)$ is *one-to-one*, from Z *onto* X, and if its *inverse,* f^{-1}*, is also continuous*, then we say that f is a homeomorphism and that the metric spaces Z and X are homeomorphic. If (Z, ρ_Z) is compact, then any continuous, one-to-one mapping f from Z onto X is a homeomorphism. A function, $f : Z \to X$ is an embedding if $f : Z \to f(Z)$ is a homeomorphism. In this case we can think of Z as a topological subspace of X by identifying Z with its image $f(Z)$.

7.2 Continua

Given metric space (Z, ρ_Z), a set $E \subseteq Z$ is *connected* if E cannot be written as the union of two disjoint open sets (or two disjoint closed sets). A set $E \subseteq Z$ is *locally connected*

[14] More detail on the topics covered in this Appendix can be found in Willard (1970) and Illanes and Nadler (1999).

at $e \in E$ if each neighborhood U_e of e *contains* a connected neighborhood V_e of e. E is locally connected if it is locally connected at each $e \in E$.[15]

If the metric space, (Z, ρ_Z), is compact and connected it is called a *continuum.* Given any continuum, (Z, ρ_Z), a point $z \in Z$ is called a cut point of Z if $Z \backslash \{z\}$ is not connected. A nonempty closed, connected subset of Z is called a *subcontinuum.* If in addition, the continuum, (Z, ρ_Z), is locally connected it is called a *Peano continuum.*

A subset, C, of metric space (Z, ρ_Z) is called an *n-cell* if it is homeomorphic to $I^n := \prod_{i=1}^n [0,1]_i := [0,1]^n$. If in particular, C is homeomorphic to the interval $[0,1]$ it is called an *arc* (i.e., an arc, then, is a 1-cell). An *end point* of arc C is either one of the two points of C that are the images of the end points of $[0,1]$ under *any* homeomorphism of $[0,1]$ onto C. A continuum Z is *arcwise connected* if any two points, z^1 and z^2, in Z can be joined by an arc in Z with endpoints z^1 and z^2.

We close this subsection by noting that in any metric space (Z, ρ_Z) the condition of being (i) a locally connected continuum (i.e., a Peano continuum) and (ii) the continuous image of an interval are equivalent (this is the Mazurkiewicz-Moore Theorem - see Kuratowski, 1972). Thus, a Peano continuum (with or without an M-convex metric) is the continuous image of the unit interval, $[0,1]$.

7.3 Homotopies

We begin by recalling the notion of a homotopy - a function that essentially provides us with a way to index a set of continuous functions.

Definition A3.1 *(Homotopies) Let $C(Z \times [0,1], X)$ denote the collection of all continuous functions, $h : Z \times [0,1] \longrightarrow X$, defined on $Z \times [0,1]$ taking values in X. A function $h \in C(Z \times [0,1], X)$ is called a homotopy and each homotopy specifies an index set of continuous functions,*

$$\mathbb{H}_h(Z, X) := \{h(\cdot, t) : t \in [0,1]\}.$$

The indexed collection, $\mathbb{H}_h(Z, X)$, can be thought of as an arc, α_h, in the continuum of continuous functions, $\mathcal{C}(Z, X)$, equipped with the sup metric. The continuous functions, f and g in $\mathcal{C}(Z, X)$ are homotopically related or homotopic, if f and g are the endpoints of an arc α_h whose arc type is identified by some function, $h \in \mathcal{C}(Z \times [0,1], X)$, called a homotopy. In particular, if $f, g \in \mathcal{C}(Z, X)$ are homotopic, then there is an arc of type $h \in \mathcal{C}(Z \times [0,1], X)$ running from continuous function $f(\cdot) = h(\cdot, 0)$ to continuous function $g(\cdot) = h(\cdot, 1)$. We denote this h-arc from f to g by writing $g \in [f]_h$ or by writing $f \stackrel{h}{\longrightarrow} g$ (and if the orientation is in the opposite direction, then we write $f \in [g]_h$ or $g \stackrel{h}{\longrightarrow} g$). Constant functions form a special class of homotopy arc end points. Let $g_{\overline{x}} \in \mathcal{C}(Z, X)$ denote the constant function (i.e., $g_{\overline{x}}(z) = \overline{x}$ for all $z \in Z$). If f and $g_{\overline{x}}$ are homotopic (i.e., if $g_{\overline{x}} \in [f]_h$, that is, if $f \stackrel{h}{\longrightarrow} g_{\overline{x}}$ for some $\overline{x} \in X$), then f is said to be *inessential.* Moreover, if for some pair of compact metric spaces, (Z, ρ_z) and (X, ρ_X), all pairs of functions, $f, g \in$

[15] Local connectedness differs from connectedness. To see this, note for example that the set E in R given by

$$E = [0,1) \cup (1,2]$$

is locally connected but not connected (because E is equal to the union of two disjoint, half open intervals). While the set G in R^2 given by

$$G := \{(x,0), (x, \frac{1}{n}) : 0 \leq x \leq 1 \text{ and } n = \pm 1, \pm 2, \ldots\} \cup \{(0,y), (1,y) : y \in R\}$$

is connected but not locally connected (because only the point $(0,0)$ and $(1,0)$ in G possess a collection of connected neighborhoods). These examples are taken from Willard (1970), Chapter 8.

$\mathcal{C}(Z,X)$, are homotopic, then in particular, $f, g_{\overline{x}} \in \mathcal{C}(Z,X)$, are homotopic for some h-arc and some $\overline{x} \in X$ - and this means that for this pair of compact metric spaces, (Z, ρ_z) and (X, ρ_X), all functions , $f \in \mathcal{C}(Z,X)$, are inessential (i.e., for each $f \in \mathcal{C}(Z,X)$, there is $(h(\cdot,\cdot), x) \in (\mathcal{C}(Z \times [0,1], X), X)$, $f \overset{h}{\longrightarrow} g_{\overline{x}}$).

7.4 AR-Spaces and ANR-Spaces

A space Z is an absolute retract, denoted $Z \in AR$, if whenever Z is embedded in some a metric space, say X, then the embedded copy, $f(Z)$, of Z in X - with homeomorphism $f : Z \to f(Z) \subset X$, is a retract of X. A space Z is an absolute neighborhood retract, denoted $Z \in ANR$, if whenever Z is embedded in some a metric space, say X, then the embedded copy, $f(Z)$, of Z in X - with homeomorphism $f : Z \to f(Z) \subset X$, is a retract of some neighborhood of $f(Z)$ in X.

7.5 Contractible Spaces

If $Z \subseteq X$, then Z is *contractible* in X if for some homotopy $h \in \mathcal{C}(Z \times [0,1], X)$, there is an h-arc running from the identity (or inclusion) mapping, $f_{id} \in \mathcal{C}(Z,X)$ to a constant mapping, $g_{\overline{x}} \in \mathcal{C}(Z,X)$, for some $\overline{x} \in X$. Thus, $f_{id}(\cdot) = h(\cdot, 0)$ where $f_{id}(z) = z$ for all $z \in Z$ is the inclusion mapping (i.e., $f_{id}(z) = z = h(z,0)$ for all $z \in Z$) and $h(\cdot, 1)$ is the constant mapping (i.e., $h(z,1) = \overline{x}$ for all $z \in Z$ for some $\overline{x} \in X$).

We say that X is contractible if X is contractible in X. Note that if X is contractible, then for any $Z \subseteq X$, Z is contractible in X. By far the most useful facts related to the contractibility of continua are the following:

(1) If X is contractible and $Z \subseteq X$ is a retraction of X, then Z is also contractible. Thus if $r : X \overset{\text{onto}}{\longrightarrow} Z$, $r \in \mathcal{C}(X,Z)$ where $r(z) = z$ for all $z \in Z$, then Z is also contractible.

(2) If X is contractible, then X is unicoherent (see Corollary A.12.10 in van Mill, 2001) - implying that all pairs of functions, $f, g \in \mathcal{C}(X, S^1)$, are homotopic, for the unit circle, $S^1 := \left\{ x = (x_1, x_2) : (x_1)^2 + (x_2)^2 = 1 \right\}$. Thus, if X is contractible, then all continuous functions, $f : X \to S^1$ are inessential and we can conclude that X contains no simple closed curves.

7.6 R_δ-Spaces

A space Z is called an R_δ-space, denoted $Z \in R_\delta$, if there exists a sequence of compact, nonempty AR spaces, $\{X^n\}_n$ such that

$$X^{n+1} \subseteq X^n \text{ for every } n$$
$$\text{and}$$
$$X = \cap_{n=1}^{\infty} X^n.$$

If Z is compact, then we have the following inclusion ordering over the topological properties of Z:

$$AR \subset \text{contractible} \subset R_\delta.$$

Note that if Z is an AR space, it is an ANR space.

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