Dynamic Equilibrium with Rare Events and Heterogeneous Epstein-Zin Investors

Georgy Chabakauri

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Keywords: heterogeneous investors, Epstein-Zin preferences, rare events, equilibrium, portfolio choice.

JEL classification: D53, G11, G12

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Abstract

We consider a general equilibrium Lucas (1978) economy with one consumption good and two heterogeneous Epstein-Zin investors. The output is subject to rare large drops or, more generally, can have non-lognormal distribution with higher cumulants. The heterogeneity in preferences generates excess stock return volatilities, procyclical price-dividend ratios and interest rates, and countercyclical market prices of risk when the elasticity of intertemporal substitution (EIS) is greater than one. Moreover, the latter results cannot be jointly replicated in a model where investors have EIS ≤ 1 or CRRA preferences. We propose new approach for deriving equilibrium, and extend the analysis to the case of heterogeneous beliefs about probabilities of rare events.

Rare unexpected large drops in aggregate output lead to significant welfare losses, and mere anticipation of such events can have significant effect on asset prices in normal times. The growing economic literature demonstrates that accounting for the effects of rare events in general equilibrium helps resolve several asset pricing puzzles [e.g., Rietz (1988); Barro (2006, 2009); Gabaix (2012), among others]. Despite the fact that the heterogeneity in preferences is a salient feature of financial markets, the literature primarily studies economies with homogeneous investors. In this paper, we demonstrate that the interaction of rare events with heterogeneity in investors' preferences is an important source of time-variation of equilibrium processes, which generates excess stock return volatility and asset prices dynamics consistent with empirical findings. Our analysis is facilitated by a new tractable approach for solving models with heterogeneous investors.

We consider a discrete time Lucas (1978) economy with one consumption good, one Lucas tree and two investors with heterogeneous Epstein-Zin preferences. The aggregate consumption is subject to rare large falls, or more generally, can have non-lognormal distribution with higher cumulants. The financial market is complete, and the investors trade in a riskless bond, shares of the Lucas tree, and insurances against rare disasters. We study conditions under which the model generates empirically plausible magnitudes and dynamics of equilibrium processes, such as procyclical interest rates and price-dividend ratios, countercyclical Shapre ratios and stock return volatilities, and excess volatilities [e.g., Shiller (1981); Schwert (1989); Ferson and Harvey (1991); Campbell and Cochrane (1999)]. We show that these dynamics only arise when investors have Epstein-Zin preferences with intertemporal elasticity of substitution EIS > 1. Settings with EIS ≤ 1 or constant relative risk aversion (CRRA) preferences give rise to counterfactual dynamics.

Asset prices dynamics in our model are similar to those in homogeneous-investor economies with time-varying probabilities of disasters [e.g., Gabaix (2012), Wachter (2013)], although the time-variation in the equilibrium processes in the current paper is endogenously induced by investor heterogeneity. Therefore, our model complements the literature by identifying additional new economic forces that help match dynamic properties of asset prices. It also allows studying the risk sharing between investors, which is absent in homogeneous-investor economies. Furthermore, the optimal asset allocation with rare events, solved in this paper, is a challenging task even in a partial equilibrium setting. Below, we summarize our main findings, the intuition, and the methodological contribution.

We compare equilibria in three economies populated by investors with different risk

aversions and with EIS > 1, EIS < 1, and CRRA preferences, respectively. First, consistent with the previous studies, we show that rare disasters generate plausible magnitudes of riskless rates and risk premia, risk premia only weakly depend on EIS and are countercyclical.[1] Furthermore, the interest rates are procyclical when investors have Epstein-Zin preferences but, counterfactually, are countercyclical in the economy with CRRA investors.

Second, we show that the price-dividend ratios are procyclical only when EIS > 1, and countercyclical otherwise. Moreover, the anticipation of disasters decreases price-dividend ratios when EIS > 1, and increases them when EIS < 1 and when investors have CRRA preferences. Using an approximate Gordon's growth formula we show that the latter properties of price-dividend ratios are determined by the dynamics of interest rates and risk premia, and by investors' EIS. The dynamics of price-dividend ratios determines the properties of stock return volatilities. We show that stocks are more volatile than dividends when EIS > 1, and less volatile than dividends otherwise. Intuitively, because stock price is the product of the price-dividend ratio and dividend, volatility increases when both change in the same direction, that is, when the price-dividend ratio is procyclical, which happens only for EIS > 1. Therefore, having Epstein-Zin preferences with EIS > 1 is critical for matching the dynamics of asset prices and their volatilities.

Third, we explore optimal portfolios and risk sharing. In the partial equilibrium, in which asset prices are taken as given, we provide a new tractable characterization of optimal consumptions and portfolios in terms of investor's wealth-consumption ratio, which satisfies a backward equation. Time-t solution of this equation is an explicit function of time-$(t + \Delta t)$ solution, where Δt is the time interval between two dates, and is found by simple backward iteration without solving non-linear equations. In the general equilibrium, we show that the more risk averse investor provides insurance to the less risk averse investor. This happens because the latter investor holds a large fraction of wealth in stocks, and hence requires insurance, especially in times when her consumption is low.[2]

Fourth, we derive closed-form solutions in the economy where investors have identical risk aversions but different EIS. We demonstrate that differences in EIS affect the interest rates but not the risk premia. Moreover, all the equilibrium processes are deterministic

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[1]Following the literature [e.g., Longstaff and Wang (2012); Gârleanu and Panageas (2014)], we call a stochastic process countercyclical (procyclical) if its innovations are negatively (positively) correlated with the innovations of the process for the aggregate consumption.

[2]Dieckmann and Gallmeyer (2005) find a similar result in a model with one logarithmic investor and one CRRA investor with risk aversion of either 2 or 0.5, as further discussed in Section 3.

functions of time. Therefore, the difference in risk aversions emerges as the main source of risk sharing between the investors. This result explains very weak dependence of risk premia on EIS in the general model, as discussed above.

Finally, we consider an extension of the model in which the investors additionally disagree about the intensity of disasters in the economy. Such disagreement may arise because this intensity is difficult to estimate due to insufficient number of observations. Making the more risk averse investor pessimistic improves the performance of the model by decreasing interest rates and increasing risk premia and stock return volatilities.

The paper proposes a new methodology for solving models with heterogeneous Epstein-Zin investors. The tractability is due to the aggregate consumption following a discrete multinomial process because this process treats normal and rare events similarly, in contrast to continuous-time processes. We further facilitate the tractability by rewriting all equilibrium processes in such a way that they resemble their continuous-time counterparts. To solve for equilibrium, we equate investors' marginal rates of substitution and derive a system of equations for investors' consumption shares, which we solve using Newton's algorithm. Then, we characterize the state price density and all the equilibrium processes as functions of these consumption shares.

There is growing economic literature on the economic effects of rare disasters. Rietz (1988) shows that the anticipation of rare disasters can explain the equity premium puzzle of Mehra and Prescott (1985). Barro (2006, 2009) argues that the sizes and the frequency of disasters in the twentieth century are sufficient to explain high equity premia and low riskless rates in a Lucas (1978) economy with homogeneous CRRA and Epstein-Zin investors, respectively. Gabaix (2012), Gourio (2012), and Wachter (2013) consider models with time-varying disaster risks and explain numerous asset pricing puzzles. Martin (2013a, 2013b) studies asset pricing with rare disasters in economies with Epstein-Zin and CRRA investors, respectively. Backus, Chernov and Zin (2014) demonstrate that jumps have powerful effects on entropy. In contrast to the above literature, we allow for heterogeneous investors, and thus generate endogenous time-variation of equilibrium processes.[3]

Ma (1993) derives conditions for the existence and uniqueness of equilibrium with heterogeneous Epstein-Zin investors. Dieckmann and Gallmeyer (2005) consider a model

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[3]Backus, Chernov, and Martin (2011) and Julliard and Ghosh (2012) provide the evidence that the probability of disasters might be below estimates in the previous literature. The probability of disasters then might not be sufficient to explain equity premia.

similar to ours, but with CRRA investors, where one investor has logarithmic preferences while the other has risk aversion of 2 or 0.5. Dieckmann (2011) considers a similar model with incomplete markets. Chen, Joslin, and Tran (2012) consider an economy where investors have heterogeneous beliefs about the intensity of disasters in the economy. They demonstrate that ignoring this heterogeneity may lead to underestimation of the disaster probabilities extracted from asset prices. Piatti (2014) studies the effects of heterogeneous beliefs about rare events in a Lucas economy with two trees and CRRA investors. Branger, Konermann, and Schlag (2014) study a long-run risk model with optimistic and pessimistic investors that have identical Epstein-Zin preferences.

The paper is also related to the literature on asset pricing with investor heterogeneity but without rare events. Most related is the work by Gârleanu and Panageas (2014), which considers an overlapping generations model with heterogeneous Epstein-Zin investors, and also demonstrates the irrelevance of heterogeneity in EIS when risk aversions are the same. Other related works include Basak (2000, 2005), Bhamra and Uppal (2014), Borovička (2012), Buss, Uppal, and Vilkov (2013), Chabakauri (2013, 2015), Chan and Kogan (2002), Detemple and Murthy (1994), Dumas (1989), Dumas and Lyasoff (2012), Gallmeyer and Hollifield (2008), Isaenko (2008), Longstaff and Wang (2012). Related works by Ait-Sahalia, Cacho-Diaz and Hurd (2009), Liu, Longstaff, and Pan (2003), among others, study optimal portfolio allocations with event risk.

The paper is organized as follows. Section 1 discusses the economic setup and defines the equilibrium. Section 2 discusses optimal consumption and portfolio choice in partial equilibrium, and then provides the characterization of equilibrium processes. In Section 3, we provide the results of calibrations, the analysis of equilibrium, and discuss the economic intuition. Section 4 extends the model to incorporate levered claims on consumption and heterogeneous beliefs. Section 5 concludes, and Appendix A provides the proofs.

1.   Economic Setup

We consider a discrete-time Lucas (1978) economy with dates $t = 0, \Delta t, 2\Delta t, \ldots, T$, one consumption good produced by an exogenous tree, and two heterogeneous investors, A and B, with Epstein-Zin preferences over consumption. At date t the tree produces $D_t \Delta t$ units of consumption good, where D_t follows a multinomial process with n states $\omega_1, \ldots, \omega_n$:

$$\Delta D_t = D_t[m_D \Delta t + \sigma_D \Delta w_t + J_D(\omega)\Delta j_t], \tag{1}$$

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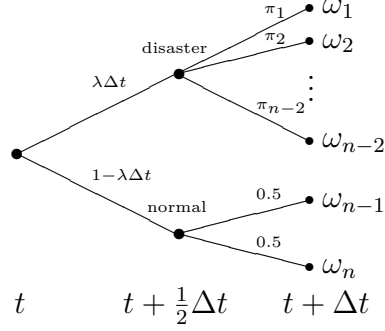
Figure 1
States of the Economy
After time t the economy moves to disaster state with small probability $\lambda\Delta t$ and to normal state with probability $1 - \lambda\Delta t$. Conditional on being in a disaster state the economy can further move to states $\omega_1, \ldots, \omega_{n-2}$ at time $t + \Delta t$ with conditional probabilities $\pi_1, \ldots, \pi_{n-2}$. Conditional on being in a normal state the economy moves to either ω_{n-1} or ω_n with equal probabilities.

where m_D and σ_D are constants, $J_D(\omega)$ is a random variable which gives the size of a drop in aggregate output following a rare disaster, and $\Delta D_t = D_{t+\Delta t} - D_t$. Parameters m_D, σ_D, $J_D(\omega)$ and Δt are such that $D_t > 0$ at all times. Processes w_t and j_t are analogues of continuous-time Brownian motion and Poisson processes and follow dynamics $w_{t+\Delta t} = w_t + \Delta w_t$ and $j_{t+\Delta t} = j_t + \Delta j_t$, respectively, where increments Δw_t and Δj_t are given by:

$$\Delta w_t = \begin{cases} 0, & \text{in states } \omega_1, \ldots \omega_{n-2}, \\ +\sqrt{\Delta t}, & \text{in state } \omega_{n-1}, \\ -\sqrt{\Delta t}, & \text{in state } \omega_n, \end{cases} \qquad \Delta j_t = \begin{cases} 1, & \text{in states } \omega_1, \ldots \omega_{n-2}, \\ 0, & \text{in state } \omega_{n-1}, \\ 0, & \text{in state } \omega_n. \end{cases} \qquad (2)$$

The structure of uncertainty is illustrated on Figure 1. From the current time-t state, the economy moves to time-$(t+\Delta t)$ disaster states with small probability $\lambda\Delta t$ or to normal states with probability $1 - \lambda\Delta t$. Disaster states $\omega_1, \ldots, \omega_{n-2}$ have conditional probabilities $\text{Prob}(\omega = \omega_k | \text{disaster}) = \pi_k$ whereas normal states ω_{n-1} and ω_n have conditional probabilities $\text{Prob}(\omega = \omega_k | \text{normal}) = 0.5$. Lemma A.1 in the Appendix shows that process (1) converges to a continuous-time Lévy process when $\Delta t \to 0$. Conveniently, $\mathbb{E}_t[\Delta w_t | \text{normal}] = 0$ and $\text{var}_t[\Delta w_t | \text{normal}] = \Delta t$, similarly to a Brownian motion, where $\mathbb{E}_t[\cdot]$ and $\text{var}_t[\cdot]$ are expectation and variance conditional on time-t information, respectively.

The discreteness of time has several advantages. First, it is more realistic to assume that investors make consumption and portfolio choice decisions discretely. Moreover, consumption data are not available at high frequencies. Second, it allows for modeling non-lognormal consumption processes with higher cumulants because such processes can be

easily approximated by multinomial dynamics. Finally, we show that the discrete-time model is more tractable, and allows passing to continuous time.

Discrete-time models with binomial or multinomial processes for consumption and asset prices are widely employed in the literature [e.g., Mehra and Prescott (1985); Rietz (1985); He (1991); Pliska (1997); Dumas and Lyasoff (2012), among others]. However, the literature typically sets $\Delta t = 1$, whereas in our calibration we take $\Delta t = 1/250$ and long horizon $T = 200$ and also take continuous-time limits in certain special cases.

1.1. Securities Markets

The financial market is complete, and the investors can trade n securities: a riskless bond in zero net supply, one stock in net supply of one unit, which is a claim to the stream of dividends $D_t \Delta t$ generated by the Lucas tree, and $n-2$ zero net supply disaster insurances each paying one unit of consumption in disaster states $\omega_1, \ldots, \omega_{n-2}$, respectively. We note that instead of using insurance contracts the market can be alternatively completed by options with different strike prices, which are written on the stock or dividends [Ross (1976)]. All trades happen at discrete dates $t = 0, \Delta t, 2\Delta t, \ldots, T$.

We consider Markovian equilibria in which bond prices, B, ex-dividend stock prices, S, and insurance prices, P_k, follow dynamics

$$\Delta B_t = B_t r_t \Delta t, \tag{3}$$

$$\Delta S_t + D_{t+\Delta t} \Delta t = S_t [m_{s,t} \Delta t + \sigma_{s,t} \Delta w_t + J_{s,t}(\omega) \Delta j_t], \tag{4}$$

$$\Delta P_{k,t} + 1_{\{\omega = \omega_k\}} = P_{k,t} [m_{P_k,t} \Delta t + \sigma_{P_k,t} \Delta w_t + J_{P_k,t}(\omega) \Delta j_t], \tag{5}$$

where $k = 1, \ldots, n-2$. Drift and volatility processes $m_{s,t}$, $\sigma_{s,t}$, $m_{P_k,t}$, and $\sigma_{P_k,t}$, and jump sizes $J_{s,t}$ and $J_{P_k,t}$ are determined in equilibrium and are adapted to time-t information.[4]

We denote the vector of drifts by $m_t = (m_{s,t}, \ldots, m_{P_{n-2},t})^\top$, the vector of risky asset expected returns by $\mu_t = (\mu_{s,t}, \mu_{P_1,t}, \ldots, \mu_{P_{n-2},t})^\top$, and the volatility matrix by $\Sigma = (\Sigma_S, \Sigma_{P_1}, \ldots, \Sigma_{P_{n-2}})^\top \in \mathbb{R}^{(n-1) \times (n-1)}$, where $\Sigma_S = (\sigma_{s,t}, J_{s,t}(\omega_1), \ldots, J_{s,t}(\omega_{n-2}))^\top$, $\Sigma_{P_k} = (\sigma_{P_k,t}, J_{P_k,t}(\omega_1), \ldots, J_{P_k,t}(\omega_{n-2}))^\top$, for $k = 2, \ldots, n-2$. We note that expected risky asset returns are given by $\mu_t = m_t + \lambda \Sigma_t (0, \pi_1, \ldots, \pi_{n-2})^\top$. Finally, we define the *state price*

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[4]We note that the dynamics of asset prices can always be written as processes (4)–(5) with Δt, Δw, and Δj terms because the vector of time-$(t+\Delta t)$ asset returns in states $\omega_1, \ldots, \omega_n$ can be uniquely decomposed as a linear combination of n linearly independent basis vectors Δt, Δw_t, and $1_{\{\omega = \omega_k\}}$, where the latter denotes an indicator function, and because $J_{s,t}(\omega) \Delta j_t = J_{s,t}(\omega_1) 1_{\{\omega = \omega_k\}} + \ldots + J_{s,t}(\omega_{n-2}) 1_{\{\omega = \omega_{n-2}\}}$.

density (SPD) ξ_t as a strictly positive process such that asset prices have representations

$$B_t = \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}B_{t+\Delta t}\Big], \tag{6}$$

$$S_t = \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\big(S_{t+\Delta t} + D_{t+\Delta t}\Delta t\big)\Big], \tag{7}$$

$$P_{k,t} = \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\big(P_{k,t+\Delta t} + 1_{\{\omega=\omega_k\}}\big)\Big], \tag{8}$$

where $S_T = 0$ and $P_{k,T} = 0$ because there are no payments after date T.

1.2. Investor Optimization

The investors have recursive utility U_t over consumption $c_{i,t}$ [e.g., Epstein and Zin (1989)], which satisfies the following backward equation[5]

$$U_{i,t} = \Big[(1 - e^{-\rho\Delta t})c_{i,t}^{1-1/\psi_i} + e^{-\rho\Delta t}\big(\mathbb{E}_t[U_{i,t+\Delta t}^{1-\gamma_i}]\big)^{\frac{1-1/\psi_i}{1-\gamma_i}}\Big]^{\frac{1}{1-1/\psi_i}}, \tag{9}$$

where $i = A, B$, γ_i and ψ_i denote investor i's risk aversion and elasticity of intertemporal substitution (EIS), respectively, and $\rho > 0$ is a time-discount parameter. In general, the investors have different risk aversions and EIS. Each period, investor i allocates fractions $\alpha_{i,t}$ and $\theta_{i,t} = (\theta_{i,S,t}, \theta_{i,P_1,t}, \ldots, \theta_{i,P_{n-2},t})^\top$ of wealth $W_{i,t}$ to riskless bonds and risky securities, respectively, so that $W_{i,t} = \alpha_{i,t}W_{i,t} + \theta_{i,t}^\top \underline{1}W_{i,t} + c_{i,t}\Delta t$, where $\underline{1} = (1, \ldots, 1)^\top \in \mathbb{R}^{n-1}$. At time 0 investor A is endowed with b units of bond, s units of stock, and p_k units of disaster insurance k, whereas investor B is endowed with $-b$ units of bond, $1 - s$ units of stock, and $-p_k$ units of insurance k. Investors solve the following dynamic programming problem [e.g., Epstein and Zin (1989)]:

$$V_{i,t} = \max_{c_{i,t},\theta_{i,t}} \Big[(1 - e^{-\rho\Delta t})c_{i,t}^{1-1/\psi_i} + e^{-\rho\Delta t}\big(\mathbb{E}_t[V_{i,t+\Delta t}^{1-\gamma_i}]\big)^{\frac{1-1/\psi_i}{1-\gamma_i}}\Big]^{\frac{1}{1-1/\psi_i}}, \tag{10}$$

where $V_{i,t}$ is investor i's value function, subject to a self-financing budget constraint[6]

$$\Delta W_{i,t} = W_{i,t}\Big(r_t + \theta_{i,t}^\top(m_t - r_t)\Big)\Delta t + W_{i,t}\theta_{i,t}^\top \Sigma_t \Delta \tilde{w}_t - c_{i,t}\Delta t(1 + r_t\Delta t), \quad W_T = c_{i,T}\Delta t, \tag{11}$$

where $i = A, B$, $\Delta\tilde{w} = (\Delta w, 1_{\{\omega=\omega_1\}}, \ldots, 1_{\{\omega=\omega_{n-2}\}})^\top$, and $1_{\{\omega=\omega_k\}}$ is an indicator function.

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[5]Gârleanu and Panageas (2014, online appendix) and Skiadas (2013) consider similar formulations with interval Δt between dates and derive continuous-time limits as $\Delta t \to 0$. The model in Skiadas (2013) allows for rare events, similarly to the present paper. Kraft and Seifried (2014) demonstrate the convergence to stochastic differential utility for Brownian risk.

[6]Investors' time t and $t + \Delta t$ wealths are given by $W_t = \alpha_t W_t + \theta_t^\top \underline{1}W_t + c_t\Delta t$ and $W_{t+\Delta t} = W_t + \alpha_t W_t \Delta B_t / B_t + \theta_{s,t}(\Delta S_t + D_{t+\Delta t}\Delta t)/S_t + \sum \theta_{P_k,t}(\Delta P_{k,t} + 1_{\{\omega=\omega_k\}})/P_{k,t}$, respectively. Substituting $\alpha_t W_t = W_t - \theta_t^\top \underline{1}W_t - c_t\Delta t$ into the latter equation and using asset price dynamics (4)–(5) we obtain budget constraint (11).

1.3. Equilibrium

Definition. *An equilibrium is a set of processes $\{r_t, \mu_t, \Sigma_t\}$ and of consumption and invest-*
ment policies $\{c_{i,t}^*, \alpha_{i,t}^*, \theta_{i,t}^*\}_{i \in \{A,B\}}$ that solve optimization problem (10) *for each investor,*
given processes $\{r_t, \mu_t, \Sigma_t\}$, and consumption and securities markets clear, that is,

$$c_{A,t}^* + c_{B,t}^* = D_t, \tag{12}$$

$$\alpha_{A,t}^* W_{A,t}^* + \alpha_{B,t}^* W_{B,t}^* = 0, \tag{13}$$

$$\theta_{A,S,t}^* W_{A,t}^* + \theta_{B,S,t}^* W_{B,t}^* = S_t, \tag{14}$$

$$\theta_{A,P_k,t}^* W_{A,t}^* + \theta_{B,P_k,t}^* W_{B,t}^* = 0, \tag{15}$$

where $k = 1, \ldots, n-2$, and $W_{A,t}^*$ and $W_{B,t}^*$ denote wealths under optimal strategies.

In addition to asset returns μ, we also study their *risk premia* $\mu - r$. We also derive
price-dividend and *wealth-consumption* ratios $\Psi = S/D$ and $\Phi_i = W_i^*/c_i^*$, respectively. We
derive a Markovian equilibrium in which the *consumption share* $y = c_B^*/D$ of investor B
is an endogenous state variable, as in the related literature [e.g., Chen, Joslin, and Tran
(2012), Gârleanu and Pedersen (2012); among others]. We demonstrate later that in a
Markovian equilibrium consumption share y_t follows a process

$$\Delta y_t = y_t[m_{y,t}\Delta t + \sigma_{y,t}\Delta w_t + J_{y,t}(\omega)\Delta j_t], \tag{16}$$

where the drift m_y, volatility σ_y, and jump sizes $J_{y,t}(\omega)$ are determined in equilibrium.

Throughout the paper, we restrict preferences and Lucas tree parameters γ_i, ψ_i, ρ, m_D,
σ_D, $J_D(\omega)$, λ, π_k, and Δt to be such that the following technical conditions are satisfied:

$$g_{i,1} \equiv e^{-\rho\Delta t} \left(\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_i}\Big] \right)^{\frac{1-1/\psi_i}{1-\gamma_i}} < 1, \tag{17}$$

$$g_{i,2} \equiv e^{-\rho\Delta t} \left(\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_i}\Big] \right)^{\frac{\gamma_i-1/\psi_i}{1-\gamma_i}} \mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma_i}\Big] < 1, \tag{18}$$

where $i = A, B$, and $g_{i,1}$ and $g_{i,2}$ are constants. We show in Section 2.2 that under
these conditions price-dividend ratio Ψ and wealth-consumption ratios Φ_i are bounded as
$T \to \infty$ in homogeneous agent economies.

2. Characterization of Equilibrium

2.1. Consumption and Portfolio Choice with Higher Cumulants

In this section, we derive optimal investment and consumption policies of investors in a partial equilibrium economy, that is, taking the asset prices dynamics (3)–(5) as given. We obtain new expressions for portfolio weights that retain the structure of their continuous-time counterparts. The advantage of our new methodology is that time-t wealth-consumption ratios $\Phi_{i,t}$ and portfolio weights $\theta_{i,t}$ are explicit functions of time-$(t + \Delta t)$ ratios $\Phi_{i,t+\Delta t}$. Therefore, all processes are obtained just by iterating explicit functions backward in time, without solving any equations.

For the time being, we do not take a stand on state variables in the economy, and assume that processes r_t, m_t, and Σ_t are functions of an unspecified Markovian variable z_t. We start by deriving closed-form discrete-time dynamics for state price density ξ_t. Lemma 1 reports the result.

Lemma 1 (State Price Density). *The state price density ξ_t follows a multinomial process*

$$\Delta\xi_t = -\frac{\xi_t}{1 + r_t\Delta t}\left[r_t\Delta t + \left(\Sigma_t^{-1}(\mu_t - r_t\underline{1})\right)^\top\left(\tfrac{1}{\Delta t}\mathrm{var}_t[\Delta\tilde{w}_t]\right)^{-1}\left(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t]\right)\right], \quad (19)$$

where $\mu_t = m_t + \Sigma_t\mathbb{E}_t[\Delta\tilde{w}_t]/\Delta t$ is the vector of expected risky asset returns, $\underline{1} = (1, \ldots, 1)^\top \in \mathbb{R}^{n-1}$, and $\Delta\tilde{w}_t$, $\mathbb{E}_t[\Delta\tilde{w}_t]$, and $\mathrm{var}_t[\Delta\tilde{w}_t]$ are given by

$$\Delta\tilde{w}_t = (\Delta w_t, 1_{\{\omega=\omega_1\}}, \ldots, 1_{\{\omega=\omega_{n-2}\}})^\top, \quad (20)$$

$$\mathbb{E}_t[\Delta\tilde{w}_t] = (0, \lambda\pi_1, \ldots, \lambda\pi_{n-2})^\top\Delta t, \quad (21)$$

$$\mathrm{var}_t[\Delta\tilde{w}_t] = \mathrm{diag}\{1 - \lambda\Delta t, \lambda\pi_1, \ldots, \lambda\pi_{n-2}\}\Delta t - \mathbb{E}_t[\Delta\tilde{w}_t]\mathbb{E}_t[\Delta\tilde{w}_t]^\top, \quad (22)$$

where $1_{\{w=w_k\}}$ is an indicator function and $\mathrm{diag}\{\ldots\}$ denotes a diagonal matrix.

The state price density process (19) preserves the structure of the familiar continuous-time process for ξ when there is no disaster risk. In particular, as in continuous-time, the drift and volatility terms of process (19) are driven by the interest rate r_t and the *market prices of risk* $\Sigma^{-1}(\mu_t - r_t)$, respectively. In a model without disasters $\Delta\tilde{w}_t = \Delta w_t$, $\mathbb{E}_t[\Delta\tilde{w}_t] = 0$, $\mathrm{var}[\Delta\tilde{w}_t] = \Delta t$, and hence from equation (19) we obtain dynamics $\Delta\xi_t = -\xi_t[r_t\Delta t + (\mu_t - r_t)/\sigma_t\Delta w_t]/(1 + r_t\Delta t)$. As $\Delta t \to 0$, the dynamics for ξ_t converges (under

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some technical assumptions) to the well-known process $d\xi_t = -\xi_t[r_t dt + (\mu_t - r_t)/\sigma_t dw_t]$ [e.g., Duffie (2001)], where μ_t and σ_t are stock mean-return and volatility, respectively. Next, we derive optimal consumptions and portfolios by solving dynamic programming problem (10). Proposition 1 reports the results.

Proposition 1 (Optimal Consumption and Investment Policies). *Investor i's time-t wealth-consumption ratio $\Phi_{i,t} = \Phi_i(z_t, t)$ is an explicit function of time-$(t + \Delta t)$ wealth-consumption ratios in states ω_n satisfying an explicit backward equation*

$$\Phi_{i,t} = e^{-\rho\psi_i\Delta t}\left(\mathbb{E}_t\left[\left(\frac{\xi_{t+\Delta t}}{\xi_t}\right)^{\frac{\gamma_i-1}{\gamma_i}}\Phi_{i,t+\Delta t}^{\frac{\gamma_i-1}{\gamma_i(1-\psi_i)}}\right]\right)^{\frac{\gamma_i(1-\psi_i)}{\gamma_i-1}} + \Delta t, \quad \Phi_{i,T} = \Delta t. \quad (23)$$

Value function $V_{i,t}$, consumption growth $c^*_{i,t+\Delta t}/c^*_{i,t}$ and portfolio $\theta^*_{i,t}$ are given by:

$$V_{i,t} = \left(\Phi_{i,t}\frac{1 - e^{-\rho\Delta t}}{\Delta t}\right)^{\frac{1}{1-1/\psi_i}}\frac{W_{i,t}}{\Phi_{i,t}}, \quad (24)$$

$$\frac{c^*_{i,t+\Delta t}}{c^*_{i,t}} = e^{-\psi_i\rho\Delta t}\left(\frac{\xi_{t+\Delta t}}{\xi_t}\right)^{-\frac{1}{\gamma_i}}\left(\Phi_{i,t+\Delta t}\right)^{\frac{\gamma_i\psi_i-1}{\gamma_i(1-\psi_i)}}\left(\mathbb{E}_t\left[\left(\frac{\xi_{t+\Delta t}}{\xi_t}\right)^{\frac{\gamma_i-1}{\gamma_i}}\left(\Phi_{i,t+\Delta t}\right)^{\frac{\gamma_i-1}{\gamma_i(1-\psi_i)}}\right]\right)^{\frac{\gamma_i\psi_i-1}{1-\gamma_i}}, \quad (25)$$

$$\theta^*_i(z_t, t) = e^{-\psi_i\rho\Delta t}(\Sigma_t^{-1})^\top\mathbb{E}_t\left[\left(\frac{\xi_{t+\Delta t}}{\xi_t}\right)^{-\frac{1}{\gamma_i}}\left(\frac{\Phi_{i,t+\Delta t}}{\Phi_{i,t}}\right)^{\frac{\gamma_i-1}{\gamma_i(1-\psi_i)}}\text{var}_t[\Delta\tilde{w}_t]^{-1}\left(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t]\right)\right]$$

$$\times\left(\mathbb{E}_t\left[\left(\frac{\xi_{t+\Delta t}}{\xi_t}\right)^{\frac{\gamma_i-1}{\gamma_i}}\left(\frac{\Phi_{i,t+\Delta t}}{\Phi_{i,t}}\right)^{\frac{\gamma_i-1}{\gamma_i(1-\psi_i)}}\right]\right)^{\frac{\gamma_i\psi_i-1}{1-\gamma_i}}, \quad (26)$$

where $i = A, B$, $\Delta\tilde{w}_t$, $\mathbb{E}_t[\Delta\tilde{w}_t]$, and $\text{var}_t[\Delta\tilde{w}_t]$ are given by equations (20)–(22) Furthermore, the state price density ξ_t is related to consumption growths $c^*_{i,t+\Delta t}/c^*_{i,t}$ as follows:

$$\frac{\xi_{t+\Delta t}}{\xi_t} = e^{-\rho\Delta t}\left(\frac{c^*_{i,t+\Delta t}}{c^*_{i,t}}\right)^{-\gamma_i}\Phi_{i,t+\Delta t}^{\frac{\gamma_i\psi_i-1}{1-\psi_i}}\left(\mathbb{E}_t\left[\left(\frac{c^*_{i,t+\Delta t}}{c^*_{i,t}}\right)^{1-\gamma_i}\Phi_{i,t+\Delta t}^{\frac{(\gamma_i-1)\psi_i}{1-\psi_i}}\right]\right)^{\frac{\gamma_i\psi_i-1}{(1-\gamma_i)\psi_i}}, \quad i = A, B, \quad (27)$$

and marginal rate of substitution $MRS_{i,t+\Delta t}(\omega_k) = \left(\partial U_{i,t}/\partial c_{i,t+\Delta t}(\omega_k)\right)/(\partial U_{i,t}/\partial c_{i,t})$ is given by $MRS_{i,t+\Delta t}(\omega_k) = \text{Prob}_t(\omega_k)\xi_{t+\Delta t}(\omega_k)/\xi_t$.

Equations (23)–(26) demonstrate that consumption and portfolio choice problem can be solved by backward induction starting from the terminal date $t = T$, if the dynamics of asset prices are known. In particular, wealth consumption ratio $\Phi_{i,t}$ is an explicit function of ratio $\Phi_{i,t+\Delta t}$ from the previous step, and hence its calculation does not require solving any equations. The wealth-consumption ratios can then be used to calculate consumption growths $c^*_{i,t+\Delta t}/c^*_{i,t}$ and portfolio weights $\theta^*_{i,t}$ using equations (25) and (26), respectively.

The equations in Proposition 1 significantly simplify for CRRA preferences. In particular, substituting $c^*_{i,t+\Delta t}/c^*_{i,t}$ from (25) into portfolio (26), after simple algebra, we obtain the following new characterization of portfolio weights

$$
\theta^*_t = \underbrace{e^{-\rho/\gamma_i \Delta t}(\Sigma^{-1}_t)^\top \operatorname{cov}_t\Big[\Big(\frac{\xi_{t+\Delta t}}{\xi_t}\Big)^{-\frac{1}{\gamma_i}}, \operatorname{var}_t[\Delta\tilde{w}_t]^{-1}\Big(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t]\Big)\Big]}_{\text{myopic demand}}
$$
$$
+ \underbrace{e^{-\rho/\gamma_i \Delta t}(\Sigma^{-1}_t)^\top \operatorname{cov}_t\Big[\frac{\Phi_{i,t+\Delta t} - \Phi_{i,t}}{\Phi_{i,t}}\Big(\frac{\xi_{t+\Delta t}}{\xi_t}\Big)^{-\frac{1}{\gamma_i}}, \operatorname{var}_t[\Delta\tilde{w}_t]^{-1}\Big(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t]\Big)\Big]}_{\text{hedging demand}}.
$$
(28)

Optimal weight (28) preserves the structure of continuous-time portfolios. In particular, the first term in equation (28) can be interpreted as myopic demand and the second term as hedging demand, as in continuous-time portfolio choice [e.g., Merton (1973); Liu (2007)]. Similar decomposition of portfolio weights can be obtained for the general case of Epstein-Zin preferences, but we do not present it for brevity.

2.2. General Equilibrium

In this section, we characterize the equilibrium. Equation (27) provides the state price density ξ_t in terms of either investor A's or investor B's consumptions. Equating the latter expressions for ξ_t and substituting in consumptions $c^*_{A,t} = (1 - y_t)D_t$ and $c^*_{B,t} = y_t D_t$, we obtain the following system of equations for finding $y_{t+\Delta t}$ as a function of y_t and state ω:

$$
\frac{\xi_{t+\Delta t}}{\xi_t} = e^{-\rho\Delta t}\Big(\frac{1 - y_{t+\Delta t}}{1 - y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma_A}\Phi_{A,t+\Delta t}^{\frac{\gamma_A\psi_A-1}{1-\psi_A}}\left(\mathbb{E}_t\Big[\Big(\frac{1 - y_{t+\Delta t}}{1 - y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_A}\Phi_{A,t+\Delta t}^{-\frac{(1-\gamma_A)\psi_A}{1-\psi_A}}\Big]\right)^{\frac{\gamma_A\psi_A-1}{(1-\gamma_A)\psi_A}}
$$
$$
= e^{-\rho\Delta t}\Big(\frac{y_{t+\Delta t}}{y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma_B}\Phi_{B,t+\Delta t}^{\frac{\gamma_B\psi_B-1}{1-\psi_B}}\left(\mathbb{E}_t\Big[\Big(\frac{y_{t+\Delta t}}{y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_B}\Phi_{B,t+\Delta t}^{-\frac{(1-\gamma_B)\psi_B}{1-\psi_B}}\Big]\right)^{\frac{\gamma_B\psi_B-1}{(1-\gamma_B)\psi_B}}. \quad (29)
$$

Intuitively, equation (29) holds because investors' marginal rates of substitution, derived in Proposition 1, are equal due to market completeness. We solve the system of equations (29) using Newton's method [e.g., Judd (1998)], and derive time-$(t + \Delta t)$ consumption shares $y_{t+\Delta t}(y_t; \omega_k)$ in states $\omega_1, \ldots, \omega_n$. Substituting shares $y_{t+\Delta t}$ back into equation (29), we obtain process $\xi_{t+\Delta t}/\xi_t$. Then, we use $\xi_{t+\Delta t}/\xi_t$ to obtain asset prices and their moments from recursive equations (6)–(8). Proposition 2 below summarizes the results.

Proposition 2 (Equilibrium Processes). *Interest rate r_t, risk premium $\mu_t - r_t\underline{1}$, price-*

dividend ratio Ψ_t and volatility Σ_t are functions of consumption share y_t, given by

$$r_t = \Big(\frac{1}{\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t]} - 1\Big)\frac{1}{\Delta t}, \tag{30}$$

$$\mu_t - r_t \underline{1} = -\frac{\Sigma_t \operatorname{cov}_t(\xi_{t+\Delta t}/\xi_t, \Delta\tilde{w}_t)}{\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t]\Delta t}, \tag{31}$$

$$\Psi_t = \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{D_{t+\Delta t}}{D_t}\Big(\Psi_{t+\Delta t} + \Delta t\Big)\Big], \quad \Psi_T = 0, \tag{32}$$

$$\Sigma_t = \mathbb{E}_t\Big[R_{t+\Delta t}\Big(\operatorname{var}_t[\Delta\tilde{w}_t]^{-1}(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])\Big)^\top\Big], \tag{33}$$

where $\Delta\tilde{w}$, $\mathbb{E}_t[\Delta\tilde{w}]$, and $\operatorname{var}_t[\Delta\tilde{w}]$ are given by equations (20)–(22), $\xi_{t+\Delta t}/\xi_t$ is given by equation (29), $\underline{1} = (1,\dots,1)^\top \in \mathbb{R}^{n-1}$ and risky assets returns $R_{t+\Delta t}$ are given by

$$R_{t+\Delta t} = \Big(\frac{\Psi_{t+\Delta t} + \Delta t}{\Psi_t}\frac{D_{t+\Delta t}}{D_t}, \frac{P_{1,t+\Delta t} + D_{1,t+\Delta t}}{P_{1,t}}, \dots, \frac{P_{n-2,t+\Delta t} + D_{n-2,t+\Delta t}}{P_{n-2,t}}\Big)^\top - \underline{1}. \tag{34}$$

Furthermore, consumption share y_t follows process (16) where $m_{y,t} = \mathbb{E}_t[\Delta y_t/y_t|\text{normal}]/\Delta t$ and $(\sigma_{y,t}, J_y(\omega_1),\dots,J_{y,t}(\omega_n))^\top = \mathbb{E}_t[y_{t+\Delta t}/y_t \operatorname{var}_t[\Delta\tilde{w}_t]^{-1}(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])]$.

To provide further intuition for the role of rare events, we obtain closed-form expressions for the equilibrium processes when investors have identical risk aversions, and when the economy is populated by homogeneous agents. When $\gamma_A = \gamma_B$ the analysis is simplified by the fact that aggregate consumption growth $D_{t+\Delta t}/D_t$ cancels out in equation (29) for consumption share $y_{t+\Delta t}$. To provide tractable expressions, we pass to continuous time limit. Proposition 3 reports the results.

Proposition 3 (Closed-Form Solutions). 1) *Suppose investors have identical risk aversions $\gamma_A = \gamma_B = \gamma$. Then, in the continuous-time limit $\Delta t \to 0$ processes r_t, $\Sigma_t^{-1}(\mu_t - r_t\underline{1})$, $\mu_{s,t} - r_t$ and $\Sigma_{s,t}$ are given by*

$$r_t = \rho + \gamma m_D - \frac{\gamma(1+\gamma)}{2}\sigma_D^2 - \lambda\Big(\mathbb{E}\big[(1 + J_D(\omega))^{-\gamma}|\text{disaster}\big] - 1\Big) \tag{35}$$

$$+ \Big(\frac{1}{\psi_B y_t + \psi_A(1 - y_t)} - \gamma\Big)\Big(m_D - \frac{\gamma}{2}\sigma_D^2 + \lambda\mathbb{E}\Big[\frac{(1 + J_D(\omega))^{1-\gamma} - 1}{1 - \gamma}\Big|\text{disaster}\Big]\Big),$$

$$\Sigma_t^{-1}(\mu_t - r_t\underline{1}) = \Big(\gamma\sigma_D, -\lambda\pi_1((1 + J_1)^{-\gamma} - 1),\dots, -\lambda\pi_{n-2}((1 + J_{n-2})^{-\gamma} - 1)\Big)^\top, \tag{36}$$

$$\mu_{s,t} - r_t = \gamma\sigma_D^2 - \lambda\mathbb{E}\big[(1 + J_D(\omega))^{-\gamma}J_D(\omega)|\text{disaster}\big] + \lambda\mathbb{E}\big[J_D(\omega)|\text{disaster}\big], \tag{37}$$

$$\Sigma_{s,t} = (\sigma_D, J_D(\omega_1),\dots, J_D(\omega_{n-2}))^\top. \tag{38}$$

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2) *In the homogeneous-investor economy with $\gamma_A = \gamma_B = \gamma$ and $\psi_A = \psi_B = \psi$, in the limit $\Delta t \to 0$ wealth-consumption ratio Ψ and insurance prices P_k are given by*

$$\Psi_t = \frac{1 - e^{-(r+(\mu_S-r)-\mu_D)(T-t)}}{r + (\mu_S - r) - \mu_D}, \quad P_{k,t} = \lambda \pi_k (1 + J_k)^{-\gamma} \frac{1 - e^{-r(T-t)}}{r}, \tag{39}$$

where r is given by equation (35) with $\psi_A = \psi_B = \psi$, $\mu_D = m_D + \lambda \mathbb{E}_t[J_D(\omega)|\text{disaster}]$ is expected dividend growth rate, and $\mu_S - r$ is given by equation (37).

Proposition 3 demonstrates that the heterogeneity in intertemporal elasticities of substitution affects only interest rates in the economy, whereas the market prices of risk, risk premia, and stock return volatility are constant and unaffected by EIS. Furthermore, in a homogeneous-investor economy with $\psi_A = \psi_B = \psi$, $\gamma_A = \gamma_B = \gamma$ and infinite horizon from Equations (35), (37) and (39) we obtain interest rate r and price-dividend ratio Ψ in Barro (2009) in the following form

$$r = \rho + \frac{1}{\psi} m_D - \frac{\gamma(1 + \psi)}{2\psi} \sigma_D^2 + \left(\frac{1}{\psi} - 1\right) \lambda \mathbb{E}_t\left[\frac{(1 + J_D)^{1-\gamma} - 1}{1 - \gamma}\Big|\text{disaster}\right] \\ + \lambda \mathbb{E}_t\left[(1 + J_D)^{-\gamma} J_D|\text{disaster}\right], \tag{40}$$

$$\Psi = \frac{1}{\rho - \left(1 - \frac{1}{\psi}\right)\left(m_D - \frac{\gamma}{2}\sigma_D^2 + \lambda \mathbb{E}_t\left[\frac{(1 + J_D)^{1-\gamma} - 1}{1 - \gamma}\Big|\text{disaster}\right]\right)}. \tag{41}$$

Equations (40) and (41) highlight the effects of EIS ψ on interest rate r and price-dividend ratio Ψ. We use the latter equations in Section 3 to facilitate the economic intuition for the results. The second term in (40) captures the consumption smoothing effect. This term decreases with higher ψ because investors with higher EIS tend to save more for consumption smoothing purposes, which pushes down the interest rates. The last three terms in (40) capture the effect of precautionary savings due to small risks and rare events. In particular, coefficient $\gamma(1 + \psi)$ in the third term measures the investor's prudence parameter for small risks Δw_t [e.g., Kimball and Weil (2009)]. The impact of the latter terms diminishes with higher ψ because the investor saves more for consumption smoothing, and hence has lower demand for precautionary savings. Finally, Equation (41) shows that the economic uncertainty, captured by volatility σ_D and disaster intensity λ, increases (decreases) price-dividend ratio for $\psi < 1$ ($\psi > 1$), as in the related literature [e.g., Bansal and Yaron (2004); Barro (2009)].

Description	Notation	Value
mean growth rate	m_D	1.8%
growth rate volatility	σ_D	3.6%
jump intensity	λ	1.7%
jump in state ω_1	$J_D(\omega_1)$	-20%
jump in state ω_2	$J_D(\omega_2)$	-30%
jump in state ω_3	$J_D(\omega_3)$	-55%
conditional probability of state ω_1	π_1	0.55
conditional probability of state ω_2	π_2	0.3
conditional probability of state ω_3	π_3	0.15
time interval	Δt	1/250
horizon	T	200

Table 1
Parameters of Aggregate Consumption Process
Dividend growth rates m_D and volatilities σ_D in normal times are from Campbell (2003); intensity λ, jump sizes $J(\omega)$ and conditional probabilities $\pi(\omega)$ are approximated from Barro (2006).

3.   Analysis of Equilibrium

In this section, we study the equilibrium processes. Figure 2 reports equilibrium interest rates r, risk premia $\mu_S - r$, price-dividend ratios Ψ, and excess volatilities $(\sigma_t - \sigma_D)/\sigma_D$ as functions of consumption share y_t in the economy without rare disasters (solid lines) and with rare disasters (dashed lines). Left, middle and right panels of Figure 2 correspond to the cases $\psi_A = \psi_B = 1.5$, $\psi_A = \psi_B = 0.5$ and CRRA preferences, respectively, for calibrated model parameters given in Table 1. In the calibration, we take dividend growth rates m_D and volatilities σ_D in normal times from Campbell (2003), and the probability of disaster λ from Barro (2006). Jump sizes $J(\omega)$ and probabilities $\pi(\omega)$ are obtained by approximating the distribution of disaster sizes in Barro (2006) by a trinomial distribution. We also fix $\Delta t = 1/250$ and $T = 200$ so that the results are not affected by the discreteness and the horizon, and set risk aversions to $\gamma_A = 3$ and $\gamma_B = 5$ and time discount to $\rho = 2\%$.

For brevity, we do not report drifts and volatilities of process (16) for consumption share y_t. We note, however, that the volatility of share y_t in normal times σ_y is negative, and hence changes Δy_t are negatively correlated with dividend changes ΔD_t, conditional on being in normal times. Intuitively, a negative shock to dividend D_t increases consumption share y of the risk averse investor B because investor A holds larger fraction of wealth in

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stocks and hence loses more wealth and consumption in bad times. Therefore, following the literature [e.g., Longstaff and Wang (2012); Gârleanu and Panageas (2014)], we call process y_t *countercyclical.* Accordingly, a process $f(y_t)$ is countercyclical (procyclical) if $f(y_t)$ is an increasing (decreasing) function of y_t.

We explore under what conditions our model can generate the equilibrium dynamics consistent with the data, such as, procyclical interest rates r and countercyclical risk premia $\mu_S - r$ [e.g., Ferson and Harvey (1991)], procyclical price-dividend ratios Ψ, countercyclical volatilities σ_t, and excess volatility $(\sigma_t - \sigma_D)/\sigma_D > 0$ [e.g., Shiller (1981); Schwert (1989); Campbell and Cochrane (1999)]. We draw two main conclusions from the results on Figure 2. First, the dynamic properties of equilibrium processes can be matched using heterogeneous preferences instead of time-varying probabilities of disasters. Second, Epstein-Zin preferences with $\psi > 1$ are crucial for generating these dynamics, and they cannot be replaced by more tractable CRRA preferences.

More specifically, with $\psi > 1$ the interest rates [Panel (A.L)] and price-dividend ratio [Panel (C.L)] are decreasing functions of share y_t, and hence are procyclical. Similarly, risk premia [Panel (B.L)] are countercyclical, stock return volatilities [Panel (D.L)] exceed dividend volatilities and are countercyclical over a wide interval of shares y, consistent with the data. In contrast, for $\psi < 1$ and CRRA preferences, counterfactually, price-dividend ratios are countercyclical [Panels (C.M) and (C.R)], volatilities are procyclical [Panels (D.M) and (D.R)], and stocks are less volatile than dividends.[7] Next, we provide the intuition for the results.

Panels (A.L), (A.M), and (A.R) of Figure 2 show the interest rates. We observe, that higher EIS decreases interest rates because investors with higher EIS save more for consumption smoothing purposes. Furthermore, the fear of rare disasters decreases the interest rates due to precautionary savings motive [e.g., Barro (2006)]. The latter effect is stronger when consumption share y_t of risk averse investor B is high, which makes rates r procyclical. In the CRRA case interest rates are countercyclical in the absence of rare events because investor B has low EIS $\psi_B = 1/\gamma_B$, and hence saves less for consumption smoothing, which offsets the precautionary savings motive.

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[7]The sizes of volatilities σ_t are significantly lower than in the data. The difficulty of matching the magnitudes of volatilities [e.g., Heaton and Lucas (1996)] is common for general equilibrium models. Because the risk premia are given by the product of Sharpe ratios and volatilities, they are also lower than in the data. To generate larger magnitudes, in Section 4.1 we following Barro (2006) and extend the model to the case of levered claims on consumption.
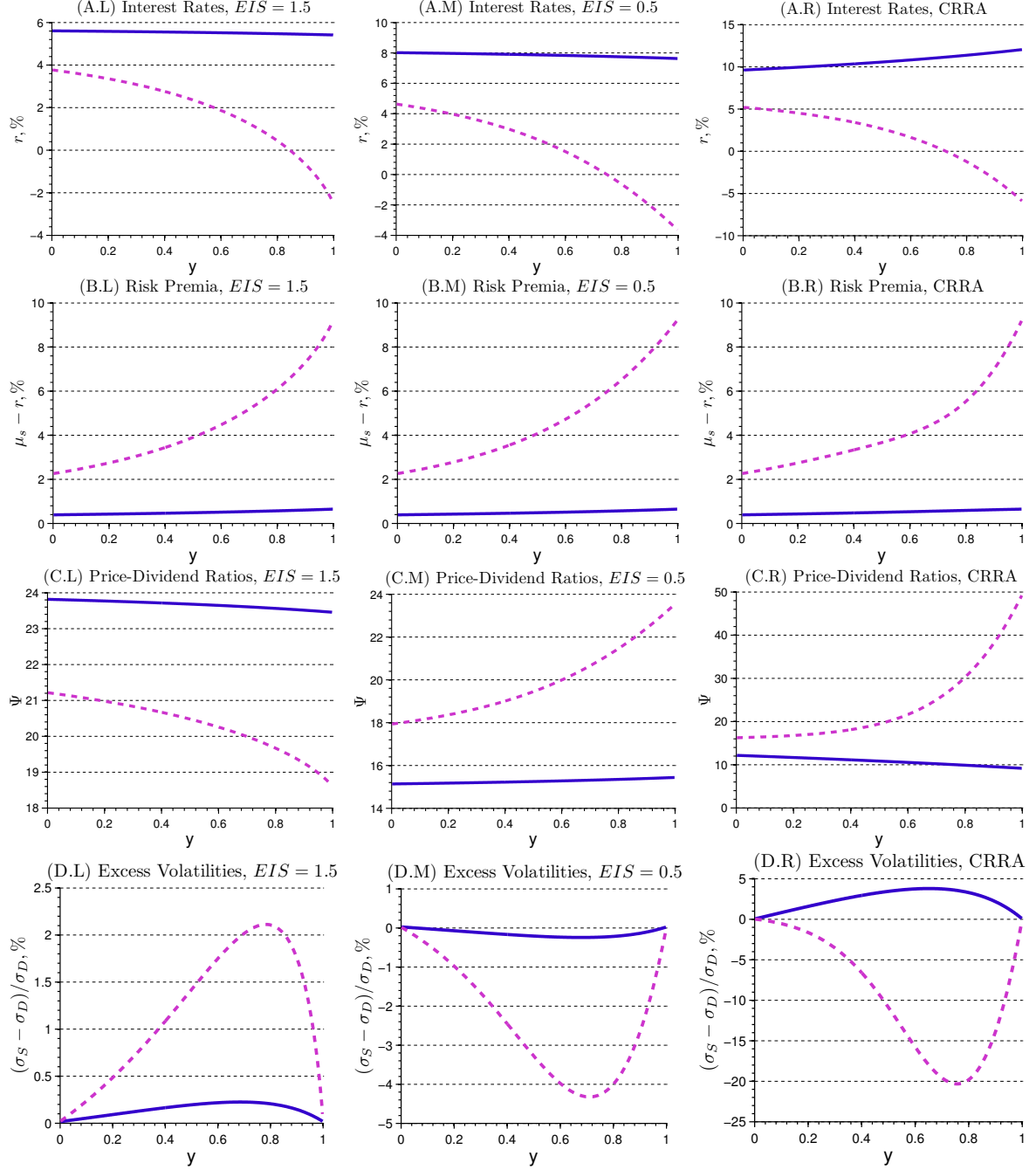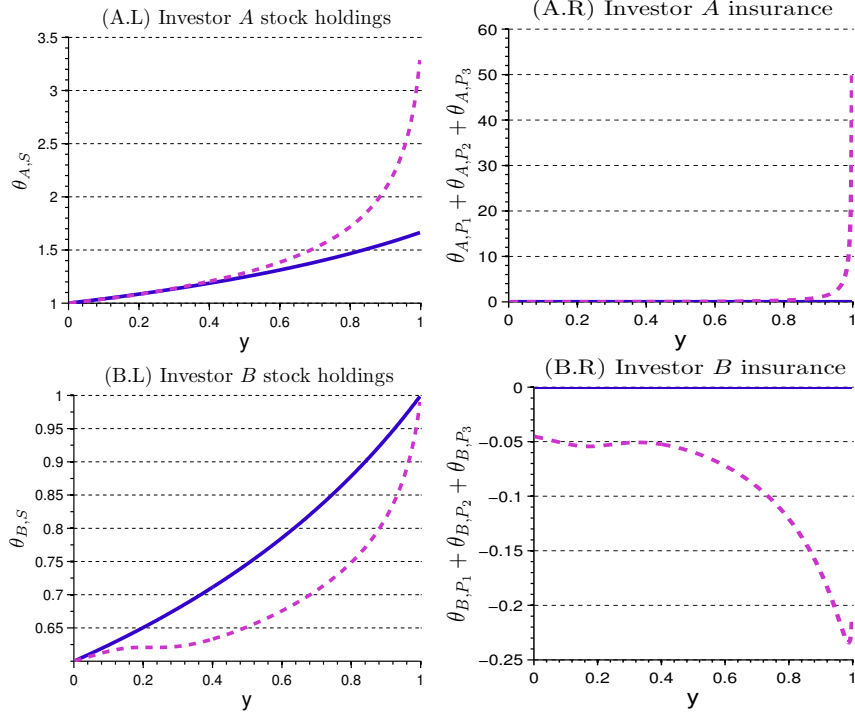
Figure 2
Equilibrium Processes

Solid (dashed) lines show the processes for economies without (with) disaster risk. Risk aversions are $\gamma_A = 3$ and $\gamma_B = 5$, and time discount parameter is $\rho = 2\%$. Left, middle, and right panels correspond to cases $\psi_A = \psi_B = 1.5$, $\psi_A = \psi_B = 0.5$, and $\psi_A = 1/\gamma_A$ and $\psi_B = 1/\gamma_B$, respectively.

Panels (B.L), (B.M), and (B.R) show risk premia $\mu_S - r$. We find that the EIS has small impact on $\mu_S - r$, consistent with the results in Section 2. Risk premia are higher when share y_t of investor B is high because investor B requires higher compensation for risk, which makes these ratios countercyclical. Moreover, the fear of rare disasters significantly increases $\mu_S - r$, bringing them in line with the estimate of 6% [e.g., Barro (2006)].

The dynamics of interest rates and risk premia and the EIS determine the dynamics of price-dividend ratios Ψ, shown on Panels (C.L), (C.M), and (C.R). In particular, we find that Gordon's growth formula $\Psi = \left(1 - \exp\{-(r + (\mu_S - r) - \mu_D)\}\right) / (r + (\mu_S - r) - \mu_D)$ derived in Proposition 3 for homogeneous-investor economies provides a good approximation also for the economies with heterogeneous investors [e.g., Chabakauri (2013, 2015)]. Therefore, counter- or pro- cyclicality of r and $\mu_S - r$ determine the properties of Ψ. In particular, Ψ is procyclical when $\psi > 1$, but countercyclical for $\psi < 1$ and CRRA preferences. We also observe that rare disasters make Ψ more procyclical (countercyclical) for $\psi > 1$ ($\psi < 1$). Moreover, the fear of disasters decreases (increases) wealth-consumption ratios when $\psi > 1$ ($\psi < 1$), consistent with equation (41) and discussion in Section 2.

Panels (D.L), (D.M), and (D.R) show the excess volatilities of stock returns over the volatilities of dividends conditional on being in normal times, $(\sigma_t - \sigma_D)/\sigma_D$. The dynamic properties of volatilities are determined by those of price-dividend ratios because stock price is given by $S_t = \Psi_t D_t$. Consequently, when $\psi > 1$, and hence Ψ_t is procyclical [see Panel (C.L)], both Ψ_t and D_t move in the same direction, which gives rise to positive excess volatility. Furthermore, volatilities turn out to be countercyclical over a large interval of consumption shares y_t. When $\psi < 1$ or the investors have CRRA preferences, ratio Ψ_t is countercyclical, and hence its variation cancels the variation in dividends, leading to lower volatility. We note that the results for total volatilities $\Sigma_{S,t}^\top \text{var}_t[\Delta \tilde{w}_t] \Sigma_{S,t}$ are qualitatively the same, and hence are not reported for brevity.

Finally, we look at portfolio weights of investors for the case of $\psi_A = \psi_B = 1.5$. Figure 3 shows the fractions of wealth that investors A and B invest in stocks [Panels (A.L) and (A.R)] and in insurance contracts [Panels (B.L) and (B.R)]. Dashed and solid lines correspond to the economies with and without rare events, respectively. The results on Figure 3 demonstrate that investor A increases the investment in stocks to take advantage of high risk premia in the economy whereas more risk averse investor B decreases the investment in stocks. The increase in the stockholding of investor A is financed by leverage, and hence $\theta_{A,S} > 1$, as in related models with heterogeneous investors [e.g., Longstaff and

Figure 3
Portfolio Strategies
Solid (dashed) lines show the processes for economies without (with) disaster risk. Risk aversions
are $\gamma_A = 3$ and $\gamma_B = 5$, EIS $\psi_A = \psi_B = 1.5$, and time discount parameter is $\rho = 2\%$.

Wang (2012); Chabakauri (2013, 2015), among others]. Investor A holds highly levered
position in risky assets when $y \approx 1$, which is made possible by very low real interest rates
[see Panel (A.L) of Figure 2].

Moreover, as investor A's consumption share decreases, that is, $1 - y_t$ goes down,
investor A increases the investment in insurance contracts. This is because when $1 - y_t$ is
low, investor A's consumption and wealth are low, and hence the investor becomes more
sensitive to disaster risk. We note that investor B allocates only a small fraction of wealth
to insurance contracts. The fraction invested by investor A in long positions in insurance
contracts is larger, but only in states where their consumption share $1 - y_t$ is small. As a
result, overall, the insurance trading has small impact on risk sharing in the economy.

Panels (B.L) and (B.R) demonstrate that, surprisingly, in the economy with rare dis-
asters more risk averse investor B sells insurance to less risk averse investor A because
the latter has very high exposure to disaster risk. Dieckmann and Gallmeyer (2005) find
a similar result in economies with two CRRA investors with risk aversions $\gamma_A = 1$ and

$\gamma_B = 0.5$ and $\gamma_A = 1$ and $\gamma_B = 2$. They demonstrate that the consumption share of the less risk averse investor is a convex function of the aggregate consumption D_t, and hence has a structure which resembles a call option. The convexity arises because investor A is overexposed to stock market, and hence has very high consumption share $1 - y_t$ when D_t is high, and very low share when D_t is low. By the put-call parity, the call is a sum of long positions in put and stock, and a short position in bond. Buying put and stock is equivalent to buying portfolio insurance. Therefore, investor A effectively buys insurance to protect consumption in bad states of the economy, where D_t is low [e.g., Dumas (1989); Dieckmann and Gallmeyer (2005)]. Without rare events the portfolio insurance is redundant and can be replicated by stock trading. However, in our economy, insurance is non-redundant, and hence has to be purchased by investor A from investor B.[8]

4. Extensions

4.1. Levered Claims on Consumption

In this section, following the literature [e.g., Barro (2006); Martin (2013a); among others], we study the risk premia and stock return volatilities of levered claims on consumption. Incorporating leverage improves the magnitude of risk premia and stock return volatilities, compared to the results in Section 3. Specifically, we consider securities with the stream of payoffs given by $D_t^\eta \Delta t$. As argued in the literature [e.g., Campbell (2003); Barro (2006); Martin (2013a)], setting $\eta > 1$ is a tractable way of capturing the effects of leverage. We use the same SPD ξ_t as in Sections 1-3 and price levered claims using backward equation

$$S_t = \mathbb{E}_t \Big[\frac{\xi_{t+\Delta t}}{\xi_t} \Big(S_{t+\Delta t} + D_{t+\Delta t}^\eta \Delta t \Big) \Big], \tag{42}$$

where $S_T = 0$, which is analogous to Equation (4) for pricing claims on consumption.

Following Barro (2006), we set $\eta = 1.5$, whereas all other exogenous parameters remain the same as in Section 3. Figure 4 shows the risk premia and return volatilities for the levered claims. Similarly to Barro (2006), we observe that incorporating leverage further increases the risk premia and stock return volatilities. However, we note that despite the large increases in volatilities, the latter remain lower than in the data.

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[8]Although in our model with Epstein-Zin preferences y_t cannot be derived as a function of consumption D_t, the intuition remains the same. The less risk averse investor A purchases insurance because, being overexposed to stock market, she has very low consumption share in bad times when D_t is low.
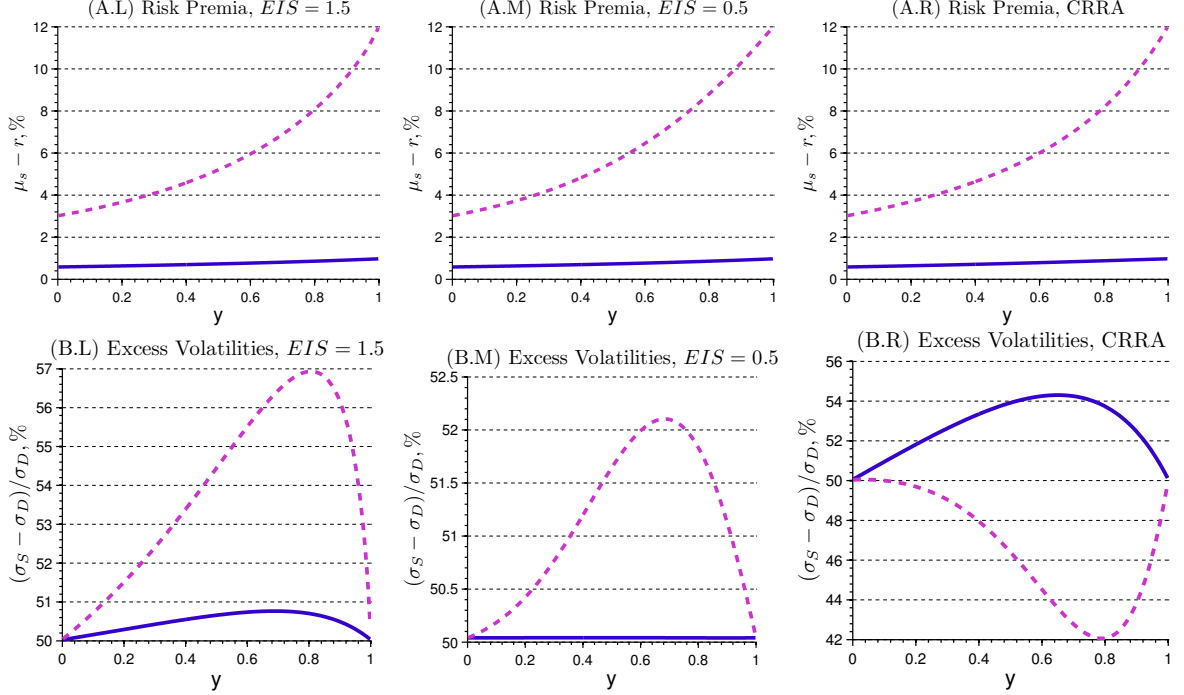
Figure 4
Risk Premia and Return Volatilities for Levered Claims on Consumption
Solid (dashed) lines show the processes for economies without (with) disaster risk. Risk aversions $\gamma_A = 3$ and $\gamma_B = 5$, and time discount parameter is $\rho = 2\%$. Left, middle, and right panels correspond to cases $\psi_A = \psi_B = 1.5$, $\psi_A = \psi_B = 0.5$, and $\psi_A = 1/\gamma_A$ and $\psi_B = 1/\gamma_B$, respectively. The claims have payoffs $D_t^\eta \Delta t$, where the leverage parameter is $\eta = 1.5$.

4.2. Heterogeneous Beliefs

Here, we study an extension of the model in Section 2 in which investors agree on observed prices and dividends but disagree on the intensity of disasters λ, because the latter are difficult to estimate due to insufficient number of observations [e.g., Chen, Joslin, Tran (2012)]. We assume that investor A has correct estimate of intensity λ, whereas investor B believes that the intensity is λ_B. For the sake of tractability, we assume that investor B does not update intensity λ_B. Proposition 4 below generalizes equation (29) to the case of heterogeneous beliefs and derives the equilibrium.

Proposition 4 (Equilibrium Processes Under Heterogeneous Beliefs). *Investors' state price densities satisfy equation $\xi_{A,t+\Delta t}/\xi_{A,t} = \eta(\omega_k)\xi_{B,t+\Delta t}/\xi_{B,t}$ in states ω_k, where $\eta(\omega_k)$ is Radon-Nikodym derivative of investor B's subjective probability measure $\mathbb{Q}$ with respect to the correct measure $\mathbb{P}$. Investor B's consumption shares $y_{t+\Delta t}(y_t; w_k)$ at time*

$t+\Delta t$ in states w_k as functions of time-t share y_t satisfy the following system of equations:

$$\Big(\frac{1-y_{t+\Delta t}(y_t;\omega_k)}{1-y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma_A}\Phi_{A,t+\Delta t}^{\frac{\gamma_A\psi_A-1}{1-\psi_A}}\left(\mathbb{E}_t\Big[\Big(\frac{1-y_{t+\Delta t}}{1-y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_A}\Phi_{A,t+\Delta t}^{-\frac{(1-\gamma_A)\psi_A}{1-\psi_A}}\Big]\right)^{\frac{\gamma_A\psi_A-1}{(1-\gamma_A)\psi_A}}=$$

$$\eta(\omega_k)\Big(\frac{y_{t+\Delta t}(y_t;\omega_k)}{y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{-\gamma_B}\Phi_{B,t+\Delta t}^{\frac{\gamma_B\psi_B-1}{1-\psi_B}}\left(\mathbb{E}_t^B\Big[\Big(\frac{y_{t+\Delta t}}{y_t}\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma_B}\Phi_{B,t+\Delta t}^{-\frac{(1-\gamma_B)\psi_B}{1-\psi_B}}\Big]\right)^{\frac{\gamma_B\psi_B-1}{(1-\gamma_B)\psi_B}},\tag{43}$$

where $\eta(\omega_1) = \ldots = \eta(\omega_{n-2}) = \lambda_B/\lambda$, $\eta(\omega_{n-1}) = \eta(\omega_n) = (1-\lambda_B)/(1-\lambda)$, and $\mathbb{E}_t^B[\cdot]$ is expectation under investor B's probability measure. Interest rate r_t, risk premia under correct beliefs $\mu_t - r_t\underline{1}$, and the volatility matrix Σ are given by equations (30)–(33), in which all expectations are under the correct beliefs of investor A, and the state price density is that of investor A, and ξ_A is given by the first equality in equation (29).

We obtain the dynamics of consumption shares $y_{t+\Delta t}$ by solving equations (43) numerically. Then, similarly to Section 3, we obtain state price density ξ_A from equation (29) and the equilibrium processes from Proposition 2. We find that making investor B pessimistic (i.e., making $\lambda_B > \lambda$) improves the performance of the model by further decreasing interest rates and increasing risk premia and volatilities. The intuition for these results can be analyzed similarly to Section 3. For brevity, we do not report the results.

5. Conclusion

This paper studies asset pricing with rare events and investor heterogeneity in a pure exchange Lucas (1978) economy. It demonstrates that EIS has significant impact on asset prices, and the model with EIS > 1 provides the best match with the data. This model generates low procyclical interest rates, large countercyclical risk premia, procyclical price-dividend ratios, and excess volatility. Moreover, the anticipation of rare events decreases price-dividend ratios and increases stock price volatilities when EIS > 1. The economies with EIS ≤ 1 or CRRA preferences generate counterfactual dynamics of asset prices.

The paper finds that the more risk averse investor provides insurance to the less risk averse investor in equilibrium, because the latter holds a very large fraction of wealth in stocks. Finally, we develop new methodology which provides tractable approach for finding optimal consumptions, portfolio strategies, and other equilibrium processes. The tractability of the solution method allows us to obtain closed-form expressions for the equilibrium processes when both investors have identical risk aversions but different EIS.

Appendix A

Lemma A.1 (Convergence of Multinomial Processes). *In the continuous time limit* $\Delta t \to 0$ *the cumulative distribution function of dividend* D_t, *which follows process (1), converges to the cumulative distribution function of a dividend following a continuous time Lévy process, given by*

$$dD_t = D_t[m_D dt + \sigma_D dw_t + J_D(\omega)\Delta j_t], \tag{A1}$$

where w_t *is a Brownian motion and* j_t *is a Poisson jump process with intensity* λ.

Proof of Lemma A.1. Consider a characteristic function $\varphi_{\Delta t}(p) = \mathbb{E}[e^{ip \ln(D_t/D_0)}]$ of random variable $\ln(D_t)$, where D_t follows process (1). Because ΔD_t are i.i.d., we obtain

$$
\begin{aligned}
\varphi_{\Delta t}(p) &= \Big(\mathbb{E}\Big[(1 + m_D \Delta t + \sigma_D \Delta w_t + J_D(\omega)\Delta j_t)^{ip}\Big]\Big)^{\frac{t}{\Delta t}} \\
&= \Big((1 - \lambda \Delta t)\mathbb{E}\Big[(1 + m_D \Delta t + \sigma_D \Delta w_t)^{ip}|\text{normal}\Big] \\
&\qquad + \lambda \Delta t\, \mathbb{E}\Big[(1 + m_D \Delta t + J_D(\omega))^{ip}|\text{disaster}\Big]\Big)^{\frac{t}{\Delta t}} \\
&= \Big(1 + ip\, m_D \Delta t + \frac{ip(ip-1)}{2}\sigma_D^2 \Delta t \\
&\qquad + \lambda \Delta t\, \mathbb{E}_t\Big[(1 + J_D(\omega))^{ip} - 1|\text{disaster}\Big] + o(\Delta t)\Big)^{\frac{t}{\Delta t}}.
\end{aligned}
\tag{A2}
$$

Taking limit $\Delta t \to 0$ we find that $\varphi_{\Delta t}(p)$ point-wise converges to function $\varphi(p)$, given by

$$\varphi(p) = \exp\Big(ip\, t m_D + \frac{ip(ip-1)}{2}t\sigma_D^2 + \lambda t\, \mathbb{E}_t[(1 + J_D(\omega))^{ip} - 1]\Big). \tag{A3}$$

Function (A3) is a characteristic function for Lévy process (A1) [e.g., Shreve (2004)]. Therefore, the distribution function for the discrete-time process D_t converges to the distribution of Lévy process (A1) by Lévy's continuity theorem [e.g., Shiryaev (1996)]. ∎

Proof of Lemma 1. Suppose, state price density follows process $\Delta \xi_t = \xi_t[a_t \Delta t + b_t^\top (\Delta \tilde{w}_t - \mathbb{E}_t[\Delta \tilde{w}_t])]$, where $\Delta \tilde{w}_t$ is given by equation (20). Next, we find coefficients a_t and b_t from the condition that equations (6)–(8) for asset prices are satisfied. The vector of time-$(t + \Delta t)$ risky asset returns can be written as $R_{t+\Delta t} = 1 + \mu_t \Delta t + \Sigma_t(\Delta \tilde{w}_t - \mathbb{E}_t[\Delta \tilde{w}_t])$, where $\mu_t = m_t + \Sigma_t \mathbb{E}_t[\Delta \tilde{w}_t]/\Delta t$ is the vector of risky assets expected returns. The equations (6)–(8) for asset prices imply that $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t B_{t+\Delta t}/B_t] = 1$ and $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t R_{t+\Delta t}] = 1$.

Substituting $B_{t+\Delta t}/B_t$ and $R_{t+\Delta t}$ into the latter equations, we obtain two equations

$$(1 + a_t\Delta t)(1 + r_t\Delta t) = 1, \tag{A4}$$

$$(1 + a_t\Delta t)(1 + \mu_t\Delta t) + \Sigma_t \operatorname{var}_t[\Delta\tilde{w}_t]b_t = 1. \tag{A5}$$

Solving equations (A4)–(A5) we obtain process (19) for the state price density. ∎

Proof of Proposition 1. Suppose, all processes are functions of a Markovian state variable z_t. The investor solves the following dynamic programming problem:

$$V_i(W_t, z_t, t) = \max_{c_{i,t}, \theta_{i,t}} \left[(1 - e^{-\rho\Delta t})c_{i,t}^{1-1/\psi_i} + e^{-\rho\Delta t}\left(\mathbb{E}_t[V_i(W_{t+\Delta t}, z_{t+\Delta t}, t+\Delta t)^{1-\gamma_i}]\right)^{\frac{1-1/\psi_i}{1-\gamma_i}} \right]^{\frac{1}{1-1/\psi_i}}. \tag{A6}$$

For simplicity, we omit subscript i for the rest of the proof. Next, we substitute $W_{t+\Delta t}$ from budget constraint (11) into optimization (A6), and taking derivatives with respect to $c_{i,t}$ and $\theta_{i,t}$ we obtain the following first order conditions:

$$e^{-\rho\Delta t}\left(\mathbb{E}_t\Big[\frac{V_{t+\Delta t}^{1-\gamma}}{V_t^{1-\gamma}}\Big]\right)^{\frac{\gamma-1/\psi}{1-\gamma}} \mathbb{E}_t\Big[\frac{\partial V_{t+\Delta t}}{\partial W_{t+\Delta t}}\frac{V_{t+\Delta t}^{-\gamma}}{V_t^{-\gamma}}(1 + r_t\Delta t)\Delta t\Big] = (1 - e^{-\rho\Delta t})\Big(\frac{V_t}{c_t}\Big)^{1/\psi}, \tag{A7}$$

$$\mathbb{E}_t\Big[\frac{\partial V_{t+\Delta t}}{\partial W_{t+\Delta t}}\frac{V_{t+\Delta t}^{-\gamma}}{V_t^{-\gamma}}\Big((m_t - r_t)\Delta t + \Sigma_t\Delta\tilde{w}_t\Big)\Big] = 0. \tag{A8}$$

To proceed further, we conjecture that $c_t^* = W_t/\Phi(z_t, t)$ and that θ_t^* does not depend on W_t, which can be verified by backward induction starting at terminal date T, where $W_T = c_T^*\Delta t$, and hence $\Phi_i(z_T, T) = \Delta t$. To find $\partial V/\partial W$, we substitute c_t^* and θ_t^* into equation (A6), and differentiating V_t in (A6) with respect to W_t we obtain:

$$\frac{\partial V_t}{\partial W_t} = V_t^{1/\psi}\left((1 - e^{-\rho\Delta t})c_t^{-1/\psi}\frac{1}{\Phi_t} + e^{-\rho\Delta t}\left(\mathbb{E}_t\big[V_{t+\Delta t}^{1-\gamma}\big]\right)^{\frac{\gamma-1/\psi}{1-\gamma}}\mathbb{E}_t\left[\frac{\partial V_{t+\Delta t}}{\partial W_{t+\Delta t}}V_{t+\Delta t}^{-\gamma}\right.\right.$$
$$\left.\left.\times \left(1 + r_t\Delta t + (\theta_t^*)^\top(m_t - r_t)\Delta t + (\theta_t^*)^\top\Sigma_t\Delta\tilde{w}_t - \frac{(1 + r_t\Delta t)\Delta t}{\Phi_t}\right)\right]\right). \tag{A9}$$

Using the first order conditions (A7)–(A8) to simplify equation (A9), we find that

$$\frac{\partial V_t}{\partial W_t} = \frac{1 - e^{-\rho\Delta t}}{\Delta t}\Big(\frac{V_t}{c_t}\Big)^{1/\psi}. \tag{A10}$$

Substituting equation (A10) back into equations (A7)–(A8), after some algebra, we obtain:

$$e^{-\rho\Delta t}\Big(\mathbb{E}_t\Big[\Big(\frac{V_{t+\Delta t}}{V_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{\gamma-1/\psi}{1-\gamma}}\mathbb{E}_t\Big[\Big(\frac{V_{t+\Delta t}}{V_t}\Big)^{1/\psi-\gamma}\Big(\frac{c_{t+\Delta t}^*}{c_t^*}\Big)^{-1/\psi}\Big(1 + r_t\Delta t\Big)\Big] = 1,$$

$$e^{-\rho\Delta t}\Big(\mathbb{E}_t\Big[\Big(\frac{V_{t+\Delta t}}{V_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{\gamma-1/\psi}{1-\gamma}}\mathbb{E}_t\Big[\Big(\frac{V_{t+\Delta t}}{V_t}\Big)^{1/\psi-\gamma}\Big(\frac{c_{t+\Delta t}^*}{c_t^*}\Big)^{-1/\psi}\Big(1 + \mu_t\Delta t + \Sigma_t\Delta\tilde{w}_t\Big)\Big] = 1.$$

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Substituting $1 + R_{t+\Delta t} = 1 + \mu_t \Delta t + \Sigma_t \Delta \tilde{w}_t$ and $B_{t+\Delta t}/B_t = 1 + r_t \Delta t$ into the latter equations, where $R_{t+\Delta t}$ is the vector of risky asset returns, and comparing the resulting equations with equations (6)–(8) for asset prices, we obtain that

$$\frac{\xi_{t+\Delta t}}{\xi_t} = e^{-\rho \Delta t}\Big(\mathbb{E}_t\big[V_{t+\Delta t}^{1-\gamma}\big]\Big)^{\frac{\gamma-1/\psi}{1-\gamma}} V_{t+\Delta t}^{1/\psi-\gamma}\Big(\frac{c_{t+\Delta t}^*}{c_t^*}\Big)^{-1/\psi}. \tag{A11}$$

Next, we prove equation (24) for the value function. Multiplying both sides of equation (A11) by $(V_{t+\Delta t})^{1-1/\psi}(c_{t+\Delta t}^*/c_t^*)^{1/\psi}$ and taking expectation $\mathbb{E}_t[\cdot]$ on both sides we obtain

$$e^{-\rho \Delta t}\Big(\mathbb{E}_t\big[V_{t+\Delta t}^{1-\gamma}\big]\Big)^{\frac{1-1/\psi}{1-\gamma}} = \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\Big(\frac{c_{t+\Delta t}^*}{c_t^*}\Big)^{1/\psi} V_{t+\Delta t}^{1-1/\psi}\Big]. \tag{A12}$$

Rewriting equation (A6) for V_t in terms of $(V_t/c_t)^{1-1/\psi}$ and using equation (A12) we find that $(V_t/c_t)^{1-1/\psi}$ solves the equation

$$\begin{aligned}
\Big(\frac{V_t}{c_t^*}\Big)^{1-1/\psi} &= 1 - e^{-\rho \Delta t} + \frac{1}{(c_t^*)^{1-1/\psi}}\Big(\mathbb{E}_t\big[V_{t+\Delta t}^{1-\gamma}\big]\Big)^{\frac{1-1/\psi}{1-\gamma}} \\
&= 1 - e^{-\rho \Delta t} + \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{c_{t+\Delta t}^*}{c_t^*}\Big(\frac{V_{t+\Delta t}}{c_{t+\Delta t}^*}\Big)^{1-1/\psi}\Big].
\end{aligned} \tag{A13}$$

Furthermore, because the market is complete, wealth W_t is given by the martingale representation $W_t = c_t \Delta t + \mathbb{E}_t[(\xi_{t+\Delta t}/\xi_t)W_{t+\Delta t}]$. Rewriting the latter equation in terms of wealth-consumption ratio $\Phi_t = W_t/c_t$ we obtain a recursive equation for Φ_t:

$$\Phi_t = \Delta t + \mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{c_{t+\Delta t}^*}{c_t^*}\Phi_{t+\Delta t}\Big]. \tag{A14}$$

Comparing the latter equation with equation (A13) we conclude that $(V_t/c_t^*)^{1-1/\psi} = (1 - e^{\rho \Delta t})\Phi_t/\Delta t$. Substituting consumption $c_t^* = W_t/\Phi_t$, after simple algebra, we obtain expression (24) for the value function. Next, substituting equation (24) for V_t into equation (A11) for state price density ξ_t, after simple algebra, we prove expression (27) for ξ_t in Proposition 1. Optimal consumption growths (25) can be obtained by solving equation (A11), which provides ξ_t in terms of $c_{t+\Delta t}^*/c_t^*$. We omit the details, but note that it can be directly verified by substitution that $c_{t+\Delta t}^*/c_t^*$ in equation (25) satisfies equation (A11). Backward equation (23) for Φ_t can be obtained by substituting $c_{t+\Delta t}^*/c_t^*$ given by equation (25) into equation (A14) for Φ_t.

It remains to prove expression (26) for θ^*. First, we rewrite budget constraint (11) under optimal strategies θ^* and c^* as $\Delta W_t = (\ldots)\Delta t + W_t(\theta_t^*)^\top \Sigma(\tilde{w}_t - \mathbb{E}_t[\Delta \tilde{w}_t])$. Multiplying both sides by $(\tilde{w}_t - \mathbb{E}_t[\Delta \tilde{w}_t])^\top$ and then taking expectations, we obtain that

$\mathbb{E}_t[(W_{t+\Delta t}/W_t)(\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top] = (\theta_t^*)^\top \Sigma_t \operatorname{var}_t[\Delta\tilde{w}_t]$. Next, replacing $W_{t+\Delta t}$ and W_t by $\Phi_{t+\Delta t}c_{t+\Delta t}^*$ and $\Phi_t c_t^*$, respectively, and solving for θ_t^* we obtain equation

$$\theta_i^*(z_t, t) = (\Sigma_t^{-1})^\top \mathbb{E}_t\Big[\frac{\Phi_{i,t+\Delta t}}{\Phi_{i,t}}\frac{c_{i,t+\Delta t}^*}{c_{i,t}^*}\operatorname{var}_t[\Delta\tilde{w}_t]^{-1}\Big(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t]\Big)\Big], \qquad (A15)$$

Substituting consumption growth $c_{i,t+\Delta t}^*/c_{i,t}^*$ from equation (25) into equation (A15) we obtain optimal portfolio weight (26) in Proposition 1.

Finally, we find $MRS_{t+\Delta t}(\omega_k) = \big(\partial U_t/\partial c_{t+\Delta t}(\omega_k)\big)/(\partial U_t/\partial c_t)$:

$$\begin{aligned} MRS_{t+\Delta t}(\omega_k) &= \frac{\partial U_t}{\partial U_{t+\Delta t}}\frac{\partial U_{t+\Delta t}/\partial c_{t+\Delta t}}{U_t/\partial c_t} \\ &= e^{-\rho\Delta t}\Big(\mathbb{E}_t\big[U_{t+\Delta t}^{1-\gamma}\big]\Big)^{\frac{\gamma - 1/\psi}{1-\gamma}}U_{t+\Delta t}^{1/\psi-\gamma}\Big(\frac{c_{t+\Delta t}}{c_t}\Big)^{-1/\psi}\operatorname{Prob}_t(\omega_k). \end{aligned} \qquad (A16)$$

Under optimal strategies θ_t^* and c_t^*, we obtain that $U_t = V_t$, and hence, from equation (A11) we obtain that $MRS_{t+\Delta t}(\omega_k) = \operatorname{Prob}_t(\omega_k)\xi_{t+\Delta t}(\omega_k)/\xi_t$. ∎

Proof of Proposition 2. Taking expectation $\mathbb{E}_t[\cdot]$ on both sides of equation (19) for ξ_t, we find that $\mathbb{E}_t[\xi_{t+\Delta t}] = 1/(1 + r_t\Delta t)$. Solving the latter equation, we obtain r_t in equation (30). Next, multiplying both sides of equation (19) by $(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top$ and taking expectations, we obtain that $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top] = -(\Sigma_t^{-1}(\mu_t - r_t\underline{1}))^\top/(1 + r_t\Delta t)/\Delta t$. Solving for $(\mu_t - r_t\underline{1})$, we obtain equation (31) for the risk premia.

To obtain Σ_t, from the dynamics of asset prices (4)–(5), we observe that asset returns $R_{t+\Delta t}$, defined by equation (34), are given by $R_{t+\Delta t} = \mu_t\Delta t + \Sigma_t(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])$. Multiplying both sides by $(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top$ and taking expectations, we obtain $\mathbb{E}_t[R_{t+\Delta t}(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])^\top] = \Sigma_t \operatorname{var}_t[\Delta\tilde{w}_t]$. Solving the latter equation, we obtain equation (33) for Σ. Next, we derive backward equation (32) for the price-dividend ratio by substituting $S_t = \Psi_t D_t$ into equation (7). Finally, we note that if equation (29) for consumption share $y_{t+\Delta t}$ has solution $y_{t+\Delta t}(y_t; \omega)$, then $\xi_{t+\Delta t}/\xi_t$ is also is a function of y_t and ω. Consequently, from equations (30)–(32) we obtain that all the equilibrium processes are functions of y_t, and returns $R_{t+\Delta t}$ are also functions of state ω. The drift and volatility of consumption share y_t are found analogously. ∎

Proof of Proposition 3. 1) From equation (29) for consumption share $y_{t+\Delta t}$, we note that when the risk aversions are the same, $\gamma_i = \gamma$, term $(D_{t+\Delta t}/D_t)^{-\gamma}$ cancels out from

the equation. Factoring out terms with $y_{t+\Delta t}$ and $\Phi_{i,t}$ from the expectation operators in equation (29) and canceling terms, we obtain that $y_{t+\Delta t}$ satisfies a deterministic equation

$$\left(\frac{y_{t+\Delta t}}{y_t}\right)^{1/\psi_B} = \left(\frac{1 - y_{t+\Delta t}}{1 - y_t}\right)^{1/\psi_A} \left(\mathbb{E}_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma}\right]\right)^{\frac{1/\psi_A - 1/\psi_B}{1-\gamma}}. \tag{A17}$$

Using similar algebra, from equation (29) we find that ξ_t is given by:

$$\frac{\xi_{t+\Delta t}}{\xi_t} = e^{-\rho\Delta t}\left(\frac{y_{t+\Delta t}}{y_t}\right)^{-1/\psi_B}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma}\left(\mathbb{E}_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma}\right]\right)^{\frac{\gamma\psi_B - 1}{(1-\gamma)\psi_B}}. \tag{A18}$$

Now, we pass to the limit $\Delta t \to 0$. First, we substitute $D_{t+\Delta t}/D_t$ from the aggregate consumption process (1) into $\mathbb{E}_t[(D_{t+\Delta t}/D_t)^\alpha]$, and obtain the following expansion:

$$\begin{aligned}
\mathbb{E}_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^\alpha\right] &= \mathbb{E}_t\left[\left(1 + m_D\Delta t + \sigma_D\Delta w_t + J_D(\omega)\Delta j_t\right)^\alpha\right] \\
&= \frac{1 - \lambda\Delta t}{2}\left[(1 + m_D\Delta t + \sigma_D\sqrt{\Delta t})^\alpha + (1 + m_D\Delta t - \sigma_D\sqrt{\Delta t})^\alpha\right] \\
&\quad + \lambda\Delta t\, \mathbb{E}\left[\left(1 + m_D\Delta t + J_D(\omega)\right)^\alpha|\text{disaster}\right] \\
&= 1 + \left(\alpha m_D + \frac{\alpha(\alpha - 1)}{2}\sigma_D^2 + \lambda\left\{\mathbb{E}_t\left[(1 + J_D(\omega))^\alpha\right] - 1\right\}\right)\Delta t + o(\Delta t).
\end{aligned} \tag{A19}$$

Next, substituting expansions $(y_{t+\Delta t}/y_t)^{1/\psi_B} = 1 + (1/\psi_B)(\Delta y_t/y_t)\Delta t + o(\Delta t)$ and $((1 - y_{t+\Delta t})/(1 - y_t))^{1/\psi_A} = 1 - (1/\psi_A)(\Delta y_t/(1 - y_t))\Delta t + o(\Delta t)$ into equation (A17), we obtain a linear equation for Δy_t. Using expansion (A19), after some algebra, we obtain expansion:

$$\Delta y_t = \frac{(\psi_B - \psi_A)y_t(1 - y_t)}{\psi_B y_t + \psi_A(1 - y_t)}\left(m_D - \frac{\gamma}{2}\sigma_D^2 + \frac{\lambda}{1 - \gamma}\left\{\mathbb{E}_t\left[\left(1 + J_D(\omega)\right)^{1-\gamma}|\text{disaster}\right] - 1\right\}\right)\Delta t + o(\Delta t). \tag{A20}$$

Using expansions (A19) and (A20), we obtain expansion for $\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t]$, where $\xi_{t+\Delta t}/\xi_t$ is given by equation (A18). Then, we derive an expansion for interest r_t, given by (30), and passing to the limit $\Delta t \to 0$, after some algebra, we obtain closed-form solution (35). The expression for the market price of risk (36) is obtained similarly, using the same expansions, and equation (31) for the risk premia in Proposition 1.

Finally, we derive the stock risk premium. Writing down the dynamics for stock prices (7) in states ω_{n-1} and ω_n, and using the fact that price-dividend ratio $\Psi_{t+\Delta t}$ is deterministic, after some algebra, we obtain expressions for the drift of the stock price:

$$1 + m_{s,t}\Delta t = \frac{\Psi_{t+\Delta t} + \Delta t}{2\Psi_t}\left(\frac{D_{t+\Delta t}(\omega_{n-1})}{D_t} + \frac{D_{t+\Delta t}(\omega_n)}{D_t}\right), \tag{A21}$$

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where $D_{t+\Delta t}(\omega_{n-1})$ and $D_{t+\Delta t}(\omega_n)$ denote time-$t + \Delta t$ dividend in states ω_{n-1} and ω_n, respectively. Moreover, from equation (30), $1 + r_t \Delta t = 1/\mathbb{E}_t[\xi_{t+\Delta t}/\xi_t]$ and from equation (32), $(\Psi_{t+\Delta t} + \Delta t)/\Psi_t = 1/\mathbb{E}_t[(\xi_{t+\Delta t}/\xi_t)(D_{t+\Delta t}/D_t)]$. Using the above equations, we find

$$m_{s,t} - r = \frac{1}{2}\Big(\frac{D_{t+\Delta t}(\omega_{n-1})}{D_t} + \frac{D_{t+\Delta t}(\omega_n)}{D_t}\Big)/\mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{D_{t+\Delta t}}{D_t}\Big] - 1/\mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\Big]. \qquad (A22)$$

Risk premium is then found as $\mu_{s,t} - r_t = m_{s,t} - r_t + \Sigma_{s,t}\mathbb{E}_t[\tilde{w}_t]/\Delta t$. We also note that because Ψ_t is deterministic, the volatility $\sigma_{s,t}$ and jump sizes $J_{s,t}(\omega)$ of stock prices are the same as those of dividend process (1). Therefore, $\Sigma_{s,t} = (\sigma_D, J_D(\omega_1), \ldots, J_D(\omega_{n-2}))^\top$. Substituting $\xi_{t+\Delta t}/\xi_t$ from equation (A18) into equation (A22) and noting from the dividend dynamics (1) that $D_{t+\Delta t}(\omega_{n-1})/D_t + D_{t+\Delta t}(\omega_n)/D_t = 2 + 2m_D \Delta t$, using expansions (A19) and (A20), after some algebra, we obtain risk premium (37).

2) Now, consider the case of homogeneous investors, that is, $\psi_A = \psi_B = \psi$, $\gamma_A = \gamma_B = \gamma$. From equation (32) for ratio Ψ_t and the fact that it is deterministic, we find

$$\begin{aligned}
\Psi_t &= (\Psi_{t+\Delta t} + \Delta t)\mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{D_{t+\Delta t}}{D_t}\Big] \\
&= (\Psi_{t+\Delta t} + \Delta t)e^{-\rho\Delta t}\Big(\mathbb{E}_t\Big[\Big(\frac{D_{t+\Delta t}}{D_t}\Big)^{1-\gamma}\Big]\Big)^{\frac{1-1/\psi}{1-\gamma}},
\end{aligned} \qquad (A23)$$

where the second equality is obtained by substituting ξ_t from equation (A18) into equation (A23) and noting that $y_{t+\Delta t} = y_t$ in homogeneous investor economy. Solving backward equation (A23) we obtain that $\Psi_t = \big(1 - g_{i,1}^{(T-t+\Delta t)/\Delta t}\big)/(1 - g_{i,1})g_{i,1}\Delta t$ where $g_{i,1}$ is given by equation (17). As $T \to \infty$, the solution converges to a stationary one iff $g_{i,1} < 1$.

Next, we obtain another representation for Ψ_t in terms of rate r and risk premium $\mu_s - r$. Using the expression for $\xi_{t+\Delta t}$ from equation (19), we obtain:

$$\begin{aligned}
\mathbb{E}_t\Big[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{D_{t+\Delta t}}{D_t}\Big] &= \frac{1}{1+r\Delta t}\mathbb{E}_t\Big[\Big(1 - (\Sigma^{-1}(\mu-r))^\top(\text{var}_t[\Delta\tilde{w}_t]/\Delta t)^{-1}(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])\Big) \\
&\qquad \times \Big(1 + m_D\Delta t + \Sigma_D^\top\mathbb{E}_t[\Delta\tilde{w}_t] + \Sigma_D^\top(\Delta\tilde{w}_t - \mathbb{E}_t[\Delta\tilde{w}_t])\Big)\Big] \\
&= \frac{1 + \Big(m_D + \Sigma_D^\top\mathbb{E}_t[\Delta\tilde{w}_t]/\Delta t - (\Sigma^{-1}(\mu-r))^\top\Sigma_D\Big)\Delta t}{1+r\Delta t},
\end{aligned} \qquad (A24)$$

where $\Sigma_D = (\sigma_D, J_D(\omega_1), \ldots, J_D(\omega_{n-2}))^\top$. Using formula (36) for $\Sigma^{-1}(\mu - r)$, after some algebra, as $\Delta t \to 0$, we obtain $(\Sigma^{-1}(\mu-r))^\top\Sigma_D = \gamma\sigma_D^2 - \lambda\mathbb{E}_t[(1 + J_D(\omega))^{-\gamma}J_D(\omega)|\text{disaster}] + \lambda\mathbb{E}_t[J_D(\omega)|\text{disaster}]$. Furthermore, it can be shown by some algebra that $\Sigma_D^\top\mathbb{E}_t[\Delta\tilde{w}_t] =$

$\lambda \Delta t \mathbb{E}_t[J_D(\omega)|\text{disaster}]$. Substituting the latter expressions into equation (A24), we obtain

$$\mathbb{E}_t\left[\frac{\xi_{t+\Delta t}}{\xi_t}\frac{D_{t+\Delta t}}{D_t}\right] = 1 - (r + (\mu_S - r) - m_D)\Delta t + o(\Delta t). \tag{A25}$$

Substituting (A25) into equation (A23) we find that in the limit $\Psi'(t) - (r + (\mu_S - r) - m_D)\Psi(t) + 1 = 0$, subject to $\Psi(T) = 0$. Solving the ODE we obtain Ψ_t in equation (39).

Similarly, given that prices $P_{k,t}$ are deterministic, from equation (8), we obtain:

$$\begin{aligned}
P_{k,t} &= P_{k,t+\Delta t}\mathbb{E}_t\left[\frac{\xi_{t+\Delta t}}{\xi_t}\right] + \lambda\pi_k\Delta t\frac{\xi_{t+\Delta t}(\omega_k)}{\xi_t} \\
&= P_{k,t+\Delta t}\frac{1}{1+r\Delta t} + \lambda\pi_k\Delta t e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}(\omega_k)}{D_t}\right)^{-\gamma}\mathbb{E}_t\left(\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma}\right]\right)^{\frac{\gamma\psi-1}{(1-\gamma)\psi}} \\
&= e^{-\rho\Delta t}P_{k,t+\Delta t}\mathbb{E}_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma}\right]\mathbb{E}_t\left(\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma}\right]\right)^{\frac{\gamma\psi-1}{(1-\gamma)\psi}} \\
&\quad + \lambda\pi_k\Delta t e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}(\omega_k)}{D_t}\right)^{-\gamma}\mathbb{E}_t\left(\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma}\right]\right)^{\frac{\gamma\psi-1}{(1-\gamma)\psi}}.
\end{aligned} \tag{A26}$$

Iterating backward it can be demonstrated that $P_{k,t} = \left(1 - g_{i,2}^{(T-t)/\Delta t}\right)/(1 - g_{i,2})b_k\lambda\Delta t$, where $g_{i,2}$ is given by equation (18), and b_k is given by:

$$b_k = \lambda\pi_k e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}(\omega_k)}{D_t}\right)^{-\gamma}\mathbb{E}_t\left(\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma}\right]\right)^{\frac{\gamma\psi-1}{(1-\gamma)\psi}}. \tag{A27}$$

Passing to continuous time limit in the second equality in equation (A26), similarly to price-dividend ratios Ψ_t, we obtain the insurance prices in equation (39). ∎

Proof of Proposition 4. Because the investors agree on observed asset prices, using equations (6)–(8) for asset prices in terms of the state price density, we obtain:

$$B_t = \mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}B_{t+\Delta t}\right] \tag{A28}$$

$$= \mathbb{E}_t^B\left[\frac{\xi_{B,t+\Delta t}}{\xi_B,t}B_{t+\Delta t}\right]$$

$$S_t = \mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}\left(S_{t+\Delta t} + D_{t+\Delta t}\Delta t\right)\right] \tag{A29}$$

$$= \mathbb{E}_t^B\left[\frac{\xi_{B,t+\Delta t}}{\xi_B,t}\left(S_{t+\Delta t} + D_{t+\Delta t}\Delta t\right)\right],$$

$$P_{k,t} = \mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}\left(P_{k,t+\Delta t} + 1_{\{\omega=\omega_k\}}\right)\right] \tag{A30}$$

$$= \mathbb{E}_t^B\left[\frac{\xi_{B,t+\Delta t}}{\xi_{B,t}}\left(P_{k,t+\Delta t} + 1_{\{\omega=\omega_k\}}\right)\right].$$

The expectations under investor B's subjective probability measure in equations (A31)–(A31) can be rewritten in terms of the expectations under the correct measure of investor A and Radon-Nikodym derivative $\eta_{t+\Delta t}(\omega)$ to obtain:

$$\mathbb{E}_t\Big[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}B_{t+\Delta t}\Big] = \mathbb{E}_t\Big[\eta_{t+\Delta t}\frac{\xi_{B,t+\Delta t}}{\xi_{B},t}B_{t+\Delta t}\Big],$$

$$\mathbb{E}_t\Big[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}\big(S_{t+\Delta t}+D_{t+\Delta t}\Delta t\big)\Big] = \mathbb{E}_t\Big[\eta_{t+\Delta t}\frac{\xi_{B,t+\Delta t}}{\xi_{B},t}\big(S_{t+\Delta t}+D_{t+\Delta t}\Delta t\big)\Big],$$

$$\mathbb{E}_t\Big[\frac{\xi_{A,t+\Delta t}}{\xi_{A,t}}\big(P_{k,t+\Delta t}+1_{\{\omega=\omega_k\}}\big)\Big] = \mathbb{E}_t\Big[\eta_{t+\Delta t}\frac{\xi_{B,t+\Delta t}}{\xi_{B,t}}\big(P_{k,t+\Delta t}+1_{\{\omega=\omega_k\}}\big)\Big].$$

From the latter equations and from the uniqueness of the state price density under the correct expectations, demonstrated in Lemma 1, we obtain that $\xi_{A,t+\Delta t}/\xi_{A,t} = \eta_{t+\Delta t}\xi_{B,t+\Delta t}/\xi_{B,t}$. Next, using the latter equality and equation (27) for the state price density in terms of investors consumptions, similarly to equation (29) we obtain a system of equations (43) for consumption shares $y_{t+\Delta t}(y_t; w_k)$. Because the time is discrete, the Radon-Nikodym derivative is simply given by the ratio of subjective investor B's and real probabilities of states $\omega_1, \dots \omega_n$. Therefore, the Radon-Nikodym derivative does not depend on time, and hence can be written as $\eta(\omega)$. ∎

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