Information Asymmetries, Volatility, Liquidity, and the Tobin Tax

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JEL classification: G12, D82.

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Information Asymmetries, Volatility, Liquidity, and the Tobin Tax*

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Abstract

We develop a tractable model in which trade is generated by asymmetry in agents' information sets. We show that, even if news are not generated by a stochastic volatility process, in the presence of information treatment and/or order processing costs, the (unique) equilibrium price process is characterised by stochastic volatility. The intuition behind this result is simple. In the presence of trading costs and dynamic information, agents strategically choose their trading times. Since new (constant volatility) information is released to the market at trading times, the price process sampled at trading times is not characterised by stochastic volatility. But since trading and calendar times differ, the price process at calendar times is the time change of the price process at trading times – i.e. price movements on the calendar time scale are characterised by stochastic volatility. Our closed form solutions show that: *i)* volatility is autocorrelated and is a non-linear function of both number and volume of trades; *ii)* the relative informativeness of numbers and volume of trades depends on the sampling frequency of the data; *iii)* volatility, the limit order book, and liquidity, in terms of tightness, depth, and resilience, are jointly determined by information asymmetries and trading costs. The model is able to rationalise a large set of empirical evidence about stock market volatility, liquidity, limit order books, and market frictions, and provides a natural laboratory for analysing the equilibrium effects of a financial transaction tax.

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1 Introduction

The recent financial turmoil has renewed the academic interest in understanding whether, and how, financial risk is endogenously generated in the marketplace. Moreover, policies intended to reduce financial market volatility, and possibly increase market liquidity, have come to the forefront of the economic and political discourse. In particular, in the form of a financial transaction tax – aka a Tobin tax – as a device for throwing sand in the wheels of the financial market. We contribute to this discourse by developing a (dynamic) equilibrium theory of financial market volatility, liquidity, and limit order book, in which *stochastic* volatility is endogenously generated (even if economic fundamentals and information flows have constant variance) by the strategic interaction of agents endowed with different information about the fundamental value of a financial asset.

In the (noisy rational expectation) equilibrium setting we consider, volatility, liquidity (in terms of tightness, depth, and resilience), and the limit order book, are jointly determined by the degree of asymmetric information and trading frictions on the market. Moreover, the equilibrium price process of the traded risky assets is characterised by self-exciting dynamics even though fundamental values are not.

Our model provides micro foundations for a large set of financial markets empirical regularities such as: *a*) the presence of time varying, and clustering, volatility for the price of risky assets; *b*) a large set of stylised facts on the link between return volatility and market volume, as well as between volatility and number of trades; *c*) the evidence that market volatility is increasing, and liquidity decreasing, in the degree of trading costs and adverse selection; *d*) the contemporaneous occurrence of volatility spikes and liquidity dry-ups; *e*) the empirical link between frequency of trading activity, price impact of trades, and the dynamics of price adjustments to new information releases.[1]

We consider an asymmetric information sequential trading framework à la Glosten and Milgrom (1985) (see also e.g. Easley and O'Hara (1987), Glosten (1989), Brunnermeier and Pedersen (2009)), with several additional novel, and salient, features. First, we allow for the endogenous determination of the volume of trade by considering a (competitive) market maker that can post a complete price schedule as a function of the order size of each trader's demand. In this formulation, the market maker can be thought of as representing the total limit order book of the market. Second, we let (informed, and less informed – aka "noisy") traders to freely choose whether and how much to trade with the market maker. Third, we consider both dynamic information and trade frictions (the latter in the form of a propor-

[1]See e.g. Gallant, Rossi, and Tauchen (1992), Jones, Kaul, and Lipson (1994), Ané and Geman (2000), Benston and Hagerman (1974), Amihud and Mendelson (1989), Keim and Madhavan (1996), Loeb (1983), Kavajecz (1999), Umlauf (1993), Hiemstra and Jones (1994), Andersen (1996), Chan and Fong (2000), Hausman, Lo, and MacKinlay (1992), Farmer and Lillo (2004), Dufour and Engle (2000), Jones and Seguin (1997).

tional trading cost i.e. analogous to a financial transaction tax and/or an order processing cost). Fourth, we relax the canonical sequential trading assumption of financial markets being observed at discrete exogenous intervals by considering a limiting market in which the potential traders arrival rate goes to infinity, hence approximating a continuously observed financial market, but with trading activity still happening at discrete – endogenously determined, yet stochastic – points in time, as in real world markets. Fifth, we characterise the equilibrium dynamics of the price process in both trade and calendar time scales, and at several frequencies (the tick-by-tick, medium, low and ultra-low frequencies), by developing a novel approach that relies on the asymptotic characterisation of the equilibrium market dynamics sampled at different frequencies and on different time scales. This allows us to show that the equilibrium has market invariance properties in the sense of Kyle and Obizhaeva (2011a, 2001b).

In the market we consider, two assets are traded: a riskless security, and a risky one with final payoff determined by a continuous stochastic process. The market is populated by three types of agents. First a (risk neutral) specialist dealer (market maker) that, at any point in time, can post a complete price schedule (for any order size) at which she stands ready to trade the risky security – i.e. she chooses the entire limit order book. The specialist does not observe directly the stochastic process driving the fundamental value of the assets, and has to infer it from the history of prices, numbers, and volume of trade. Second, there is a continuum with unit mass of (market order) traders that sequentially arrive to the market according to a *weakly* exogenous stochastic counting process[2] (characterised by an arrival intensity parameter that we will be sending to infinity in order to approximate a continuously observed market). The (risk neutral) market order traders are of two types. A fraction q of them is of the uninformed (noisy trader) type, while $1 - q$ of them observe directly the continuous stochastic process determining the fundamental value of the risky asset. The share of uninformed traders, as well as agents' preferences and all the past history of trade price, time, and volume, are common knowledge.

Upon arrival, a trader observes the price schedule posted by the market maker and, based on her valuation of the asset, decides whether to trade, and how much, at the posted prices. If a trade occurs, the market maker updates her valuation of the asset based on the information that can be inferred from the market order posted by the last trader (i.e. the trader's valuation and the posterior probability of her being of the informed type) and, consequently, she updates the bid and ask pricing schedules. Like in real world markets, the market maker observes the trader's arrival if and only if the trader decides to trade (i.e. she

[2] We require the arrival process to be independent only conditional on the information revealed by the last trade, i.e. we require only weak exogeneity of the arrival process rather than the strong exogeneity (or process independence) commonly assumed in sequential trading models. This implies that, in principle, the arrival process could depend on the past trading history.

does not observe directly the arrival process) and does not know whether a trader is of the informed or noisy type (hence she has to form posterior beliefs about the trader's type).

In order to introduce a trading friction in this market, we assume that a (small) proportional trading cost is associated with each trade (as e.g. in Stambaugh (2014)). Without loss of generality, we assume that this trading cost is incurred by the market maker. Alternatively, we could have modelled the friction in the form of a minimum order size, and this would have preserved all the key equilibrium mechanics. Nevertheless, the proportional trade cost formulation has two important advantages. First, it is analogous to a Tobin Tax for financial transactions, hence it allows us to study the equilibrium effect of such a tax. Second, it makes the theoretical predictions of our model comparable with the empirical literature that has extensively modelled and estimated transaction cost specifications of this form.

The presence of a trade friction is crucial in order to generate a bid-ask spread for small orders: without such friction the limit of the ask and bid (equilibrium) price functions, as the order size approaches zero, would be identical i.e. there would be no a bid-ask spread in the proximity of zero even though the market maker faces an adverse selection problem when trading with informed traders.[3] Note that the equilibrium bid-ask spread is, as one would expect, increasing in the degree of adverse selection faced by the market maker. In turn, the bid-ask spread is crucial for endogenously generating time varying volatility. The reason behind this mechanism is quite intuitive in our settings. Prices are, in equilibrium, a mapping from the market maker's valuation process of the asset to the real line. Therefore, for asset returns to exibit heteroscedasticity, one needs the conditional and unconditional distributions of information, revealed by the trading activity, to be different. The bid-ask spread delivers this by generating an inertia region for an informed trader since, whenever her valuation is within the bid-ask spread, she optimally decides not to trade. As a consequence, the pool of information incorporated into prices changes depending of whether informed agents are in the inertia region or not.

The above implies that, changing the bid and ask price schedules, the market maker changes the distribution of information incorporated into prices. Moreover, since the market maker, upon receiving an order, never knows for certainty whether the trader is informed or uninformed, her evaluation of the asset evolves gradually (and stochastically, since it is "noised up" by both the noisy traders' activity and the continuous processes driving the fundamental value). This in turn generates an equilibrium price process that is autocorrelated and that shows stochastic clustering of volatility.

A natural way of forming intuition about the equilibrium dynamics of the model is to

[3] In Glosten and Milgrom (1985) the friction that, in the presence of adverse selection, generates the bid-ask spread, is the assumption of a fixed order size for all trades.

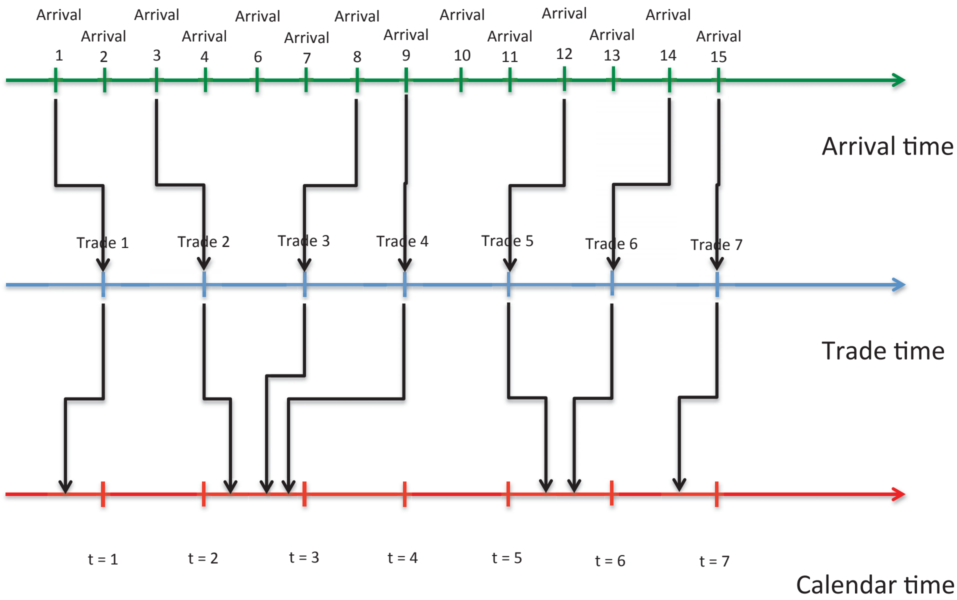

Figure 1: Market time scales.

consider the three different time scales underlying our market. These are depicted in Figure 1. The first (uppermost) time scale is the arrival time one, on which potential traders arrive to the market and observe the limit order book posted by the market maker. Upon arrival, based on their valuation and the current available price schedules, traders decide whether to trade or not. Trades then occur sequentially on the trade time scale (the middle one in the figure). If they decide to trade, agents reveal their own valuation of the asset via the order size they post, and this information gets incorporated into prices and into the updated bid and ask price schedules that the market maker posts. Note that, on the trade time scale, prices are adapted to overall information process in the market. Therefore, if there is no stochastic and clustering volatility in the fundamental information process, there won't be stochastic and clustering volatility on the trade time scale. Nevertheless, on the calendar time scale, due to the traders' endogenous decision of whether to trade or not upon arrival, the price process will be a *time change* of the process on the trade by trade time scale – i.e. price movements on the calendar time scale are characterised by stochastic volatility, due to the clustering of information revealed by the trading process.[4]

We show that, at the tick-by-tick (high) frequency, price movements and volatility are driven (in a non-linear fashion) by the (equilibrium) volume of trade process. This result is quite intuitive since, at very high frequency (i.e. trade by trade) the market maker's

[4]For the representation of a price process with stochastic volatility via time change (aka time deformation) see e.g. Mandelbrot and Taylor (1967), Clark (1973), Tauchen and Pitts (1983), Yor, Madan, and Geman (2002), Andersen, Bollerslev, and Dobrev (2007).

valuation update (hence the information that is incorporated into prices) is driven by the order size posted by traders. Moreover, the link between price movements and volume, as well as the closed form relationship between these quantities that we obtain, are consistent with the empirical findings of a large body of literature.[5]

Considering the sequence of market equilibria as the (possibly time varying) intensity of traders' arrival approaches infinity, we identify what we refer to as the *medium* frequency equilibrium price process. This is the frequency at which the market is close to being continuously observed by potential traders. Obviously, in the real world, this frequency will be asset specific (e.g., in a given calendar time interval, blue chip stocks are closer to being continuously observed by traders than a stock at the bottom of the NYSE market capitalisation distribution), and will be driven by the stock specific characteristic *business time*. That is, sending the arrival intensity to infinity we identify an equilibrium price process that is of the *market microstructure invariance* type (see e.g. Kyle and Obizhaeva (2011a, 2001b)). More precisely, financial assets with the same level of transaction costs, asymmetric information, fundamental volatility and drift will have the same equilibrium price process distributions at medium frequency. Nevertheless, what this frequency will correspond to in calendar time (hours, days, moths, etc.) will be asset specific and will depend upon the level of market attention dedicated to the assets.

At this medium frequency the trade by trade volatility is increasing in both the level of transaction costs and the degree of adverse selection faced by the market maker. This is due to the fact that, as these market frictions increase, market tightness and resilience reduce. The first effect reduces the amount of trading (via reduced liquidity and increased no trade region) while the second makes large departures from fundamental values more likely and persistent. These effects imply that, when informed traders choose to trade, price corrections are more severe. From a Tobin Tax perspective, this result implies that such a tax: *a*) increases trade by trade variance overall, and its effect is more severe in markets with a high level of adverse selection; *b*) reduces volatility in periods of small shocks to the fundamental value (i.e. in tranquil times), since conditional on small shocks the market will be more often in the no trade region; *c*) substantially increases volatility in hectic periods i.e. when large shocks to fundamental values occur.

Even though, as our sequential framework approaches a continuously observed market, the trade by trade volatility becomes constant, the calendar time scale volatility is time varying in a stochastic manner. Intuitively, this is due to the fact that, as depicted in Figure

[5]See e.g. Gallant, Rossi, and Tauchen (1992) that, using a non linear specification, find a strong link between volume of trade and price movements, as well as Farmer and Lillo (2004) and Farmer, Lillo, and Mantegna (2003), that identify a log-linear relationship between gross price growth and changes in volume, and Potters and Bouchaud (2003), that identify a log-log relationship between gross price growth and volume changes. We show that in our framework all these relationships between price growth and volume can arise in equilibrium depending on the market's fundamental characteristics.

1, trades in the calendar time scale are endogenously clustered. This intuition is confirmed by our (asymptotic) closed form solution that shows that, at low frequency (i.e. the frequency characterised by a large number of trades per time interval), the stochastic volatility of the price process is driven by the number of trades process, and this dependency is exactly of the type identified empirically by Ané and Geman (2000).

Our framework also delivers a (closed form) equilibrium characterisation of the drivers of market liquidity in terms of tightness, deepness, and resilience. In particular, we find that, as the degree of adverse selection increases, tightness is reduced, market impact increases (for small order sizes), and departures of the price from the fundamental value are expected to last longer. Moreover, since volatility (on all time scales), increases with adverse selection, our framework can rationalise the joint occurrence of liquidity dry-ups and volatility spikes (as e.g. during the subprime crises).

Given the ability of our model to rationalise several salient features of asset price dynamics – such as the empirical link between volatility and volume and number of trades, the common dynamics of volatility and liquidity, as well as the relationship between market frictions and trading activity – it constitutes a natural laboratory for analysing the equilibrium effects of the introduction of a Tobin Tax. On this front, we show that the introduction of a Tobin tax has strong effects on both volatility and liquidity. In particular, our model predicts, consistently with the empirical literature,[6] that such a tax would substantially reduce liquidity (in terms of tightness and resilience), increase volatility, and slow down the business clock of the market. Furthermore, these effects are stronger in markets characterised by a high degree of adverse selection – i.e. the effect of a Tobin Tax would be more dramatic in already illiquid and highly volatile markets. Moreover, we show that such a tax would reduce volatility in "good times" (i.e. when only small shocks to fundamental are realised) and increase volatility in "hectic times" (i.e. when large fundamental shocks occur).

More broadly, our work is also related to the large literature on information aggregation in financial market and noisy rational expectation equilibria (see e.g. Grossman and Stiglitz (1980), Hellwig (1980), Admati (1985), Kyle (1985), and Wang (1993, 1994), Easley and O'Hara (1987, 2004)). In particular, since we study a sequence of market equilibria as the intensity of potential traders arrival goes to infinity, our work is related to Back and Baruch (2004) that studies the limiting behaviour of a Glosten and Milgrom (1985) type model as uninformed trades become smaller and arrive more frequently, while the (single) insider chooses optimally when to trade. In this setting, the authors show, there is an equilibrium in which informed and uninformed traders arrive probabilistically, as we assume. Our model departs from their setting in that we do not restrict the order size to be constant (and shrinking to zero as the intensity of arrivals goes to infinity). This allows us to obtain, in

[6]See e.g. Umlauf (1993), Jones, Kaul, and Lipson (1994), Jones and Seguin (1997), and Hau (2006).

closed form, the equilibrium order book and market depth as a function of the order size. Moreover, we consider dynamic information and characterise the behaviour of the equilibrium price and volatility processes at different frequencies.

Furthermore, given our focus on the role of financial frictions and transaction taxes, our work is closely connected, respectively, to Vayanos and Wang (2012), and Subrahmanyam (1998) and Buss, Uppal, and Vilkov (2014).

The reminder of the paper is organised as follows. Section 2 introduces the trading and information structure of the market, as well as the agents' optimisation problems. Section 3 solves for the market equilibrium and characterises the resulting properties of the order book, the optimal trading behaviours, and the prices process on different time scales and at various frequencies. In Section 4 we analyse the equilibrium properties of market liquidity and volatility, while Section 5 concludes. Additional proofs and technical results are reported in the Appendix.

2 Model Primitives

All random variables are defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \leq T}, \mathbb{P}\right)$ satisfying the usual conditions. A remark about the notation used throughout the paper is worth making at this point. Since we are dealing with three different time scales – calendar time, number of arrival time, and number of trade time – processes need to be defined accordingly. For all processes we follow the convention that: i) upper case Latin letters, such as X_t, denote processes considered on the *calendar* time scale; ii) lower case Latin letters, such as x_i, denote processes considered on the *number of arrivals* time scale, that is $x_i = X_{\theta_i}$, where θ_i denotes the stopping time of the arrival process (i.e. the i-th arrival time); iii) lower case Latin letters with $\tilde{}$ superscript, such as $\tilde{x}_i$, denote processes considered on the *number of trade* time scale, that is $\tilde{x}_i = X_{\tau_i}$, where τ_i denotes the stopping time of the trade process (i.e. the i-th trade time).

2.1 Market Structure

We consider a finite trading horizon T. There are two assets: a riskless bond that yields the instantaneous return r, and a risky asset – a stock – with final value given by e^{D_T} where D is the continuous log profit process of the firm and follows the diffusion process

$$dD_t = \mu dt + \sigma dW_t^d, \quad D_0 = const,$$

where W^d is a Brownian Motion with respect to $(\mathcal{F}_t)$, and μ and σ are, respectively, the drift and volatility parameters. Note that the framework considered in this paper can be easily extended to allow for time varying μ and σ, and/or allow D to represent a best estimate, rather than the true process.

The risky asset is traded in a competitive specialist ("market maker") market. The trading structure is a sequential one as in Glosten and Milgrom (1985). Traders arrive to the market and meet the specialist according to a stochastic counting process, N, with associated stopping times $\theta_i = \inf\{t \geq 0 : N_t = i\}$ where θ_i is the time of the i-th arrival. We assume that the total number of arrivals is finite, and that future arrivals are independent from the past events.[7] We will refer to this assumption as:

A1. $N_T < \infty$ a.s. and $\sigma\{N_{\theta_i+t} - N_{\theta_i}, t \geq 0\} \perp \mathcal{F}_{\theta_i}$ for all i.

When the trader arrives to the market at time θ_i, she observes bid, $B_{\theta_i}(\cdot)$, and ask, $A_{\theta_i}(\cdot)$, prices per-share posted by the specialist. We allow the bid and ask prices per-share to depend on the order size (v). The specialist is allowed to change bid prices, $B_t(v^-)$ (where $v^- \in \mathbb{R}_+$ is the sell order size) and ask prices, $A_t(v^+)$ (where $v^+ \in \mathbb{R}_+$ is the buy order size), at any point *except* at the time at which the trader arrives. That is, as in real markets, the ask and bid quotes posted by the market maker constitute a non renegotiable trading commitment at the time at which traders decide to trade.

Given the above formulation, the market maker's posted prices can also be interpreted as orders place by competitive limit order traders, and the posted price schedules $B_t(v^-)$ and $A_t(v^+)$ can be interpreted as the time t limit order book.

We assume that the market maker has to incur a (small) proportional order processing cost for each transaction, δ. That is, if at time t the trader submitted the order to buy v^+ (or order to sell v^-) then the market maker would receive $v^+A_t(v^+)(1-\delta)$ (or spend the amount $v^-B_t(v^-)(1+\delta)$).

After observing the posted bid and ask prices, the trader that arrived at time θ_i has to decide her order size, v_i. Obviously, the trader can choose an order size of zero – in which case no trade occurs, and the specialist does not observe the i-th arrival. That is, as in the real world, the market maker will observe only the trades and not the arrivals of traders per se. The cumulative number of realised trades by time t defines the stochastic counting process

$$L_t = \sum_{i=1}^{\infty} \mathbf{1}_{\{\theta_i \leq t\} \cap \{v_i \neq 0\}}$$

where $\mathbf{1}_{\{.\}}$ is the indicator function defined over a set. We define the stopping time associated with the number of trade process, L, as $\tau_i = \inf\{t \geq 0 : L_t = i\}$ – that is, τ_i is the time of

[7] Additional assumptions on the N_t process will be outlined later. For instance, a Poisson process (with constant or time varying intensity) would satisfy these assumptions.

the i-th trade. Similarly, we define the cumulative volume of trade by time t as

$$V_t = \sum_{i=1}^{\infty} v_i \mathbf{1}_{\{\theta_i \leq t\}}. \tag{1}$$

Let $\tilde{v}_i$ indicate the order size of the i-th trade that is:[8] $\tilde{v}_i = \sum_{j=1}^{\infty} v_j \mathbf{1}_{\{\theta_j = \tau_i\}}$. Since trades always have to happen either at the bid price, $B_{\tau_i}(\cdot)$, or at the ask price, $A_{\tau_i}(\cdot)$, the price at which the i-th trade is executed is given by[9]

$$\tilde{p}_i = A_{\tau_i}\left(\tilde{v}_i^+\right) \mathbf{1}_{\{\tilde{v}_i > 0\}} + B_{\tau_i}\left(\tilde{v}_i^-\right) \mathbf{1}_{\{\tilde{v}_i < 0\}}, \tag{2}$$

since the trade has to occur either at the ask or at the bid price, and the price at time t is given by

$$P_t = \tilde{p}_{\max\{i \geq 1:\, \tau_i \leq t\}} \mathbf{1}_{\{\tau_1 \leq t\}} + \tilde{p}_0 \mathbf{1}_{\{\tau_1 > t\}}. \tag{3}$$

Note that the above formulation of P_t is needed to accommodate the case of no trades before time t, and $\tilde{p}_0$ is an equilibrium price that we will derive below.

2.2 Information Structure

Beside the specialist – the "market maker" – there are two types of traders: informed ones and uninformed noisy traders. Jointly, informed and noisy traders constitute a continuum with unit mass, are assumed to act competitively. The informational advantage of the first group is that it observes directly the D process.

To characterise the different information sets we introduce the following notation: for any given process X, we denote by $\mathcal{F}_t^X = \sigma\{X_s,\ s \leq t\} \vee \mathcal{N}$, where $\sigma\{.\}$ is the sigma algebra generated by its argument, $\mathcal{N}$ is the set of $\mathbb{P}$-null sets, and $x \vee y$ indicates the minimum sigma algebra generated by the union of x and y.

At time t all the agents observe: a) all the past history of market prices (that is the filtration $\mathcal{F}_t^P$ generated by the price process P up to time t), and b) all the past history of the cumulative volume (that is the filtration $\mathcal{F}_t^V$ generated by the volume process V). This implies that the cumulated number of trade at time t, L_t, is also known to all the market participant since it is equal to the number of jumps of $\{V_s\}_{s \leq t}$. We denote this common knowledge filtration as

$$\mathcal{G}_t^M = \mathcal{F}_t^P \vee \mathcal{F}_t^V.$$

We use the superscript M to denote the fact that $\mathcal{G}_t^M$ is also the information set of the specialist market maker.

[8]Note that $V_t = \sum_{i=1}^{\infty} \tilde{v}_i 1_{\{\tau_i \leq t\}}$.
[9]Recall that, by definition, $\tilde{v}_i \neq 0$.

For future convenience, we also define the market maker's information set at the time of the i-th trade: $\tilde{\mathcal{H}}_i^M = \mathcal{G}_{\tau_i}^M$. Note that through the paper we use the letter $\mathcal{G}$ to denote information sets in calendar time, the letter $\tilde{\mathcal{H}}$ to denote information sets in trading time, and $\mathcal{H}$ to denote information sets in the arrival time scale (i.e. $\mathcal{H}_i = \mathcal{G}_{\theta_i}^M$).

The trader who arrived at time θ_i is of the (uninformed) noisy type (U) with probability q and of the informed type (I) with probability $1-q$. We define the cumulative number of informed and uninformed traders arrival processes (N^I and N^U) and associated stopping times as (θ_i^I and θ_i^U), respectively, as

$$N_t^U = \sum_{i=1}^{\infty} \mathbf{1}_{\{\theta_i \le t\} \cap \{U_i\}}, \quad \theta_i^U = \inf\left\{t \ge 0 : N_t^U = i\right\}$$

and

$$N_t^I = N_t - N_t^U = \sum_{i=1}^{\infty} \mathbf{1}_{\{\theta_i \le t\} \cap \{I_i\}}, \quad \theta_i^I = \inf\left\{t \ge 0 : N_t^I = i\right\}$$

where U_i (I_i) denotes the event of the time θ_i trader being of the uninformed (informed) type.[10]

Since the informed trader also observes the process D , her information set upon arrival (time θ_i^I) is $\mathcal{H}_i^I = \mathcal{G}_{\theta_i^I}^{I,i}$, where $\mathcal{G}_t^{I,i} = \mathcal{G}_t^M \vee \mathcal{F}_t^D \vee \sigma\left\{\theta_i^I \wedge s, s \le t\right\}$, and $\wedge$ denotes the minimal element.

The noisy traders demand is parametrized indirectly, through their information set. In particular, we assume that, in addition to observing the market filtration $\mathcal{G}_t^M$ at time t, noisy traders receive a private signal S_t. That is, upon arrival at time θ_i^U, the noisy trader receive the private signal $s_i = S_{\theta_i^U}$ and has therefore the information set $\mathcal{H}_i^U = \mathcal{G}_{\theta_i^U}^{U,i} \vee \sigma\left\{s_i\right\}$, where $\mathcal{G}_{\theta_i^U}^{U,i} = \mathcal{G}_t^M \vee \sigma\left\{\theta_i^U \wedge s, s \le t\right\}$. This indirect modelling of the noisy traders demand simplifies exposition because: a) since the market maker will, in equilibrium, filter the information of each trader's demand, we are defining the noisy traders demand in the relevant domain for the filtering problem (rather than having to invert what a particular noisy demand schedule would imply in terms of filtered information from the market maker point of view); b) as we will show below, the requirement of noisy and informed traders' demands being indistinguishable given the market maker information set, can be very easily formulated using this indirect modelling of the noise traders' demand.

In what follows, we postulate that the following assumptions are satisfied:

A2. $\mathcal{F}_T^W, \mathcal{F}_T^N$ and S_{θ_i} are conditionally independent given $\mathcal{H}_{i-1}$for all i, where $\mathcal{H}_i = \mathcal{G}_{\theta_i}$, $\mathcal{G}_t = \mathcal{F}_t^V \vee \mathcal{F}_t^N$.

A3. I_i is independent of $\mathcal{F}_T^{N,S,D} \vee \sigma\left(U_k\right)_{k \ne i}$.

[10] Obviously $U_i \cup I_i = \Omega$.

A4. $\mathbb{P}(v_i \in C|\mathcal{H}_{i-1}, I_i, \theta_i) = \mathbb{P}(v_i \in C|\mathcal{H}_{i-1}, U_i, \theta_i)$ for $C \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra.

Assumption A2 has the following implications. First, that – as typically assumed in sequential trading models (see O'Hara (1995) for a discussion) – trader's arrival time is exogenous and independent from movements in the fundamentals. This, as in Glosten and Milgrom (1985), rules out a *strategic timing* of agents arrival but, differently from them, does not rule out all other dimensions of strategic behavior since the traders will be free to chose whether to trade or not and their order size. Note also that the only restriction that Assumption A2 imposes on the arrival process is that it should be *conditionally* independent – that is, it could in principle depend on the past trading history (summarized by $\mathcal{H}_i$). Second, $A2$ implies that the signal received by noisy traders does not carry more information about the fundamental than what could be inferred from the current history of past order sizes and arrival times.

Remark 1 *Note that $\mathcal{H}_i$ is the σ-algebra generated by $\{v_j\}_{j=0}^{i}$ and $\{\theta_j\}_{j=0}^{i}$.*

Informed traders can potentially benefit from any departures of the stock price from the fundamental value, and so informed traders could decide to trade as much as possible – but such a behaviour would quickly reveal the information of the informed to the market maker. Assumption A3 prevents this from happening by not allowing informed traders to decide when to arrive to the market. This can be seen as imposing an equilibrium behaviour, as the one studied in Easley and O'Hara (1987), in which informed agents mimic uninformed agents behaviour to avoid detection.

Jointly, assumptions A2 and A3 guarantee that the *actual* population of traders that the market maker faces is always the same as the *potential* population of traders, since none of the traders can endogenously decide when to arrive to the market.

Assumption A4 restricts the signal process received by uninformed traders. It imposes that the distribution of order size submitted by the investor (conditionally on lagged information) is independent of the type of trader, therefore guaranteeing that informed traders are inconspicuous, in the sense that they cannot be detected by the market maker. This basically imposes a "pooled" equilibrium, as the one discussed in Easley and O'Hara (1987), in which informed agents optimally decide to be pooled together with the uninformed ones. We will show later that this is equivalent to the requirement that the uninformed traders valuations of the assets do not excessively deviate from the fundamental value of the asset.

Note that the above assumptions on the signal received by the uninformed investor do not imply that these agents can only act in a purely noisy fashion. For example, it is easy to show that a setting in which noisy traders receive a noisy estimate of D_t would satisfy the above assumptions.

We also assume that agents' preferences, to be presented below, are common knowledge.

2.3 Agents' Preferences

2.3.1 Traders' Preferences

The preferences of all agents are common knowledge. Since there is a continuum of potential traders and the arrival process is exogenous, upon arrival the conditional probability of experiencing a second arrival is zero. Therefore, an agent that arrives to the market at time θ_i faces a basically static problem. Nevertheless, using the standard utility maximisation approach, a closed form solution for the market equilibrium can be provided only under the assumption of risk neutrality. As a consequence, as it is customary in the market microstructure models of information based trading, we will assume that all agents are risk neutral.

Recall that the final payoff of holding v^+ shares is simply $v^+ e^{D_T}$. Assuming that traders are risk neutral and that the inter-temporal discount factor is equal to the risk-free interest and both are equal to zero,[11] the expected utility from holding v^+ shares until time T for an agent of type $k \in \{I, U\}$ that arrived to the market at time θ_i^k is

$$\mathbb{E}\left[v^+ e^{D_T} \middle| \mathcal{H}_i^k \right] =: v^+ z_i^k \tag{4}$$

where z_i^k is the expected utility from owning one stock for a type k trader. Moreover, under the above assumptions, the expected utility from investing in the risk free asset the amount needed to buy v^+ shares at time θ_i^k is simply

$$v^+ A_{\theta_i^k}\left(v^+\right). \tag{5}$$

The expected utilities in equations (4) and (5) can be viewed as the outcome of two alternative investment strategies – buying v^+ stocks or investing $v^+ A_{\theta_i^k}\left(v^+\right)$ in the risk free asset. Since a similar expression is associated with sell orders, v^-, the optimisation problem of the agent of type k that arrives at time θ_i^k can be expressed as

$$\max_{v^+, v^-} \quad v^+ \left[z_i^k - A_{\theta_i^k}\left(v^+\right) \right] + v^- \left[B_{\theta_i^k}\left(v^-\right) - z_i^k \right]. \tag{6}$$

Note that in the above expression the first term refers to buying the stock while the second refers to selling the stock. As we will show later, in equilibrium it will never be optimal for the agent to choose both v^+ and v^- different from zero, that is the agent will either buy,

[11] Generalizing our results to allow the inter-temporal discount and the risk-free rate to be non-zero and different form each other is straightforward, we don't do so in order to simplify the exposition.

sell, or not trade.

For later usage, let define z_i as the expected value of holding one share of the asset for the agent that arrives at time θ_i, that is

$$z_i = \mathbf{1}_{\{I_i\}} z_i^I + \mathbf{1}_{\{U_i\}} z_i^U. \tag{7}$$

2.3.2 The Specialist's Preferences

We complete the model assuming the presence of a specialist market maker. The market maker faces a small proportional cost, δ, to execute the orders placed by traders. That is, if a trader at time t submits the buying order v^+ at the posted ask price $A_t(v^+)$, the market maker will receive, upon completion of the transaction, the amount $v^+ A_t(v^+)(1-\delta)$. Similarly, for executing a selling order of size v^- the specialist would face a cost of[12] $v^- B_t(v^-)(1+\delta)$. Assuming that it is the trader that incurs a transaction cost is without loss of generality: we could have attributed the transaction cost to the traders without changing the equilibrium dynamics of the model. The presence of a small transaction cost is necessary to generate interesting dynamics since, in equilibrium, this will generate a bid ask spread even in proximity of the zero order size, that is $\lim_{v\downarrow 0} A_t(v) - \lim_{v\uparrow 0} B_t(v) > 0$. This implies that the weakly exogenous (and unobservable) process of arrivals of traders, N, and the endogenous (and observable) counting process of trades, L, will not necessarily coincide.

This also implies that, in a given time interval, the difference in *number* of arrivals and trades will carry relevant information for the market maker. Nevertheless, the market maker cannot observe N_t nor, generally, to infer it from the observed number of trades. For example, if the exogenous arrival process is characterised by time varying intensity, an observed increase in the number of trades can be attributed either to *a*) a change in the intensity of the arrival process or to *b*) the fact that the market maker's estimate of the true value is incorrect and more informed traders choose to trade at the posted prices. We therefore are in need of specifying the market marker's prior beliefs about the connection between N_t and L_t. We assume that the market maker believes that $N_t = L_t\ \forall t$. This choice makes the problem tractable and has the advantage that, from the market maker's point of view, this belief is unfalsifiable under the assumption of unobservable time varying intensity of the arrival process. Moreover, this has the advantage of focusing the equilibrium market dynamic on the market maker's filtering of the agents information instead that around the filtering of the arrival process.[13]

[12] It is straightforward, in our setting, to allow for a different (and time varying) transaction costs for ask and bid orders. However, we focus on the constant symmetric cost case to simplify exposition.

[13] Note that preventing the market maker from inferring information contained in the time between arrivals is also the approach taken by Glosten and Milgrom (1985). For a framework in which the market maker learns from the time between trades, and hence updates quotes continuously, see Back and Baruch (2004).

The market maker, as the traders, is risk neutral, implying that her utility form owning one share of the stock until time T is

$$Z_t^M = \mathbb{E}\left[e^{D_T} \middle| \mathcal{G}_t^M, N_t = L_t \right]. \tag{8}$$

As in Glosten and Milgrom (1985), the specialist sets up bid and ask prices under a zero utility gain constraint – that is, the market can be thought of as being populated by a continuum of competitive (in Bertrand's sense) market makers. This assumption implies two restrictions of the market maker's behaviour. First, as in a competitive market, carrying out a trade at the posted price will not deliver a utility gain to the specialist (in her filtration). Second, the specialist should not regret, ex post, having executed the trade at the posted price. That is, if a trader submits an order of size v, the market maker utility should not decrease after carrying out the order. More precisely, the time t bid and ask prices, as a function of the order size v, must satisfy the following conditions

$$A_t\left(v^+\right)(1-\delta) = \sum_{i=1}^{\infty} \mathbf{1}_{\{i=1+L_{t-}\}}\, \mathbb{E}\left[e^{D_T} | \tilde{\mathcal{H}}_i^M, N_{\tau_i} = L_{\tau_i} \right]\Big|_{\tilde{v}_i = v^+, \tau_i = t}, \tag{9}$$

$$B_t\left(v^-\right)(1+\delta) = \sum_{i=1}^{\infty} \mathbf{1}_{\{i=1+L_{t-}\}}\, \mathbb{E}\left[e^{D_T} | \tilde{\mathcal{H}}_i^M, N_{\tau_i} = L_{\tau_i} \right]\Big|_{\tilde{v}_i = -v^-, \tau_i = t}. \tag{10}$$

Note that in the above summations there is only one non zero value of the index function for any realisation of history, implying that these expressions are simple certainty equivalence conditions that bid and ask prices must satisfy.

Remark 2 *Since the market maker sets ask and bid as a function of the volume, and the price of a transaction can only be at either the ask or the bid, we have that once the volume, V_t, is observed, the transaction price has no residual information content. That is $\mathcal{G}_t^M = \mathcal{F}_t^V \vee \mathcal{F}_t^L$, implying that $\tilde{\mathcal{H}}_i^M = \sigma\left\{ \{\tilde{v}\}_{j=0}^i, \{\tau_j\}_{j=0}^i \right\}$.*

Additionally, we impose the following regularity conditions on bid and ask functions:

C1. For a fixed v, the processes $B\left(v^-\right)$ and $A\left(v^+\right)$ are left continuous with right limits.

C2. For a fixed t, $A_t\left(v^+\right) : \mathbb{R}_+ \to \bar{\mathbb{R}}_+ \setminus \{0\}$ is continuous, nondecreasing and $\lim_{v^+ \to \infty} A_t(v^+) = +\infty$.

C3. For a fixed t, $B_t\left(v^-\right) : \mathbb{R}_+ \to \bar{\mathbb{R}}_+$ is continuous, non increasing and $\lim_{v^- \to \infty} B_t(v^-) = 0$.

C4. For a fixed t, $A_t(0) \geq B_t(0)$ for all $\omega \in \Omega$.

C5. For any fixed t, $A_t(\cdot)$ continuously differentiable, and $B_t(\cdot)$ is continuously differentiable on the set $\{v : B_t(v) > 0\}$

C6. For a fixed t, $vA_t(v)$ is strictly convex, and $vB_t(v)$ is strictly concave on the set $\{v : B_t(v) > 0\}$

Condition C1 formalises the idea that, as in the real world, the specialist can change the bid and ask functions at any point in time except at the time at which the trade occurs.

Condition C2 for the ask price implies that: *i*) the specialist will never dispose of the assets for free; *ii*) the price per-share at which the specialist will agree to sell will not decrease in the order size; *iii*) the specialist will refuse to trade infinite quantities. The first two implications are meant to match the real world ask price behaviour, while the last one rules out degenerated cases. Condition C3 for the bid price per-share is the analog of condition C2 for the ask price.

Condition C4 is a technical one, and is meant to rule out the degenerate case of ask prices being below the bid price, while Condition C5 simply assume that the bid and ask function are sufficiently smooth.

Condition C6 ensures that the traders' demand functions are uniquely determined by their valuations (i.e. it ensures strict concavity of the traders' objective function in equation (6)). This is equivalent to imposing a single crossing condition for the demand and supply functions of the asset.

Also, in order to avoid the degenerated case of no trade ever occurring due to a systematically too large bid ask spread, we require the transaction cost δ to be sufficiently small. In particular, we have the following condition.

A5. $\delta \in (0, q)$.

The connection, in the above condition, between the maximum size of the transaction cost, δ, and the share of uninformed agents, q, is intuitive. The market maker will make profits, on average, only when dealing with uformed agents. Therefore, if the transaction cost that the market maker faces is too large, relative to the share of uformed agents in the economy, it will not be optimal for her to trade and she will choose an infinite bid ask spread.

3 Market Equilibrium

3.1 Existence and uniqueness of the equilibrium

In what follows we prove existence and uniqueness of the equilibrium. We define a market equilibrium as follows.

Definition 1 (Equilibrium) *A market equilibrium is a set of policy functions* $A_t(v^+)$, $B_t(v^-)$, $v_i(A_{\theta_i}(v^+), B_{\theta_i}(v^-))$ *such that:*

1. *Conditions C1-C6 are satisfied;*
2. $A_t(v^+)$ *and* $B_t(v^-)$ *solve the specialist optimisation problem characterised by equations (9) and (10) for any* v, t*;*
3. $v_i(A_{\theta_i}(v^+), B_{\theta_i}(v^-))$ *solves the trader's problem in equation (6).*

To prove existence and uniqueness of the market equilibrium, it is first useful to establish two intermediate results. The first Lemma states the solution of the trader's optimisation problem for any ask and bid prices that satisfy conditions C1-C6.

Lemma 1 (Trader's optimal demand) *Suppose* $A_t(v^+)$, $B_t(v^-)$ *satisfy conditions* ***C1-C6****. Consider a trader who arrives on the market at time* θ_i *and observes the posted prices* $A_{\theta_i}(v^+)$ *and* $B_{\theta_i}(v^-)$*. Then*

- *if* $z_i > A_{\theta_i}(0)$*, the optimal order size,* v^**, is strictly positive and is the unique solution of*
$$z_i = A_{\theta_i}(v) + vA'_{\theta_i}(v) \tag{11}$$
- *if* $z_i < B_{\theta_i}(0)$*, the optimal order size,* v^**, is strictly negative and is the unique solution of*
$$z_i = B_{\theta_i}(-v) - vB'_{\theta_i}(-v) \tag{12}$$
- *if* $B_{\theta_i}(0) \leq z_i \leq A_{\theta_i}(0)$*, then the optimal order size is* $v^* = 0$*.*

Proof of Lemma 1. Follows from the first order conditions of the trader's problem in equation (6) and the observation that conditions C2 and C3 ensure existence and finiteness of the global maximum, while condition C6 ensures uniqueness of the maximum. Moreover, condition C4 rules out different cases from the ones considered in the Lemma. ■

It is important to stress that the trader's expected utility from owning the stock is revealed upon submission of the order v^*. This allows us to solve the filtering problem of the market maker.

The above result allows us to make an important remark on Assumption A4.

Remark 3 (Remark on Assumption A4.) *Note that the optimality conditions in Lemma 1 identify a one to one correspondence between the order size,* v_i*, and the agent's valuation,*

z_i. Denote this invertible map by $f : v_i \to z_i$. Hence, for $C \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}(z_i \in C|\mathcal{H}_{i-1}, I_i, \theta_i) = \mathbb{P}(f(v_i) \in C|\mathcal{H}_{i-1}, I_i, \theta_i) = \mathbb{P}\left(v_i \in f^{-1}C|\mathcal{H}_{i-1}, I_i, \theta_i\right),$$
$$\mathbb{P}(z_i \in C|\mathcal{H}_{i-1}, U_i, \theta_i) = \mathbb{P}(f(v_i) \in C|\mathcal{H}_{i-1}, U_i, \theta_i) = \mathbb{P}\left(v_i \in f^{-1}C|\mathcal{H}_{i-1}, U_i, \theta_i\right).$$

Therefore, assumption A4 is equivalent to $\mathbb{P}(z_i \in C|\mathcal{H}_{i-1}, U_i, \theta_i) = \mathbb{P}(z_i \in C|\mathcal{H}_{i-1}, I_i, \theta_i)$.

The above reformulation makes clear that Assumption A4 is a requirement on the type of information that the uniformed agents receives. In a nutshell, it requires that the uninformed traders' valuations of the asset do not excessively deviate from the fundamental value of the asset that is observed by the informed agent.

In the next proposition we characterise the optimal ask and bid price function from the market maker's standpoint.

Proposition 4 (Optimal ask and bid functions) *Suppose assumptions A1-A5 are satisfied. Then there exist optimal ask, $A_t(v^+)$, and bid, $B_t(v^-)$, prices that satisfy conditions C1-C5 and the market maker's optimality conditions (9) and (10). Moreover, the optimal $A_t(v)$ and $B_t(v)$ have the following forms:*

$$A_t^*(v) = \frac{q}{q-\delta}\left(1 + \alpha v^{\frac{q-\delta}{1-q}}\right)\sum_{i=0}^{\infty} \mathbf{1}_{\{i=L_{t-}+1\}} Z_{\tau_{i-1}}^M \tag{13}$$

$$B_t^*(v) = \begin{cases} \frac{q}{q+\delta}\left(1 - \beta v^{\frac{q+\delta}{1-q}}\right)\sum_{i=0}^{\infty} \mathbf{1}_{\{i=L_{t-}+1\}} Z_{\tau_{i-1}}^M & \text{if } \beta v^{\frac{q+\delta}{1-q}} \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where α and β are strictly positive arbitrary constants, and Z_t^M is given in equation (8).

Proof. The proof, being technical, is reported in Appendix A.1. Nevertheless, the steps of the proof are quite intuitive. First, we show that, in the market maker filtration, the probability of a trader being of the uninformed type is simply q independently from the order size. Second, from the order size and Lemma 1, the market maker can recover the asset valuation of the trader. Third, combining the probability of trader types, and the valuations corresponding to each order size, together with the market maker's indifference conditions (9) and (10), give rise to an ordinary differential equation (ODE) for the ask price function, and one for the bid price function. Each of these ODEs admits two solutions, but only one solution per equation satisfies conditions C2 and C3. ■

The equilibrium bid and ask price function, depicted in Figure 2 for different values of q, have important implication for market liquidity in terms of depth and tightness. These properties are discussed in detail in section 4.2. One thing to notice in the figure is that, overall, as q – the share of noisy agents – increases, the bid-ask curves become steeper (for

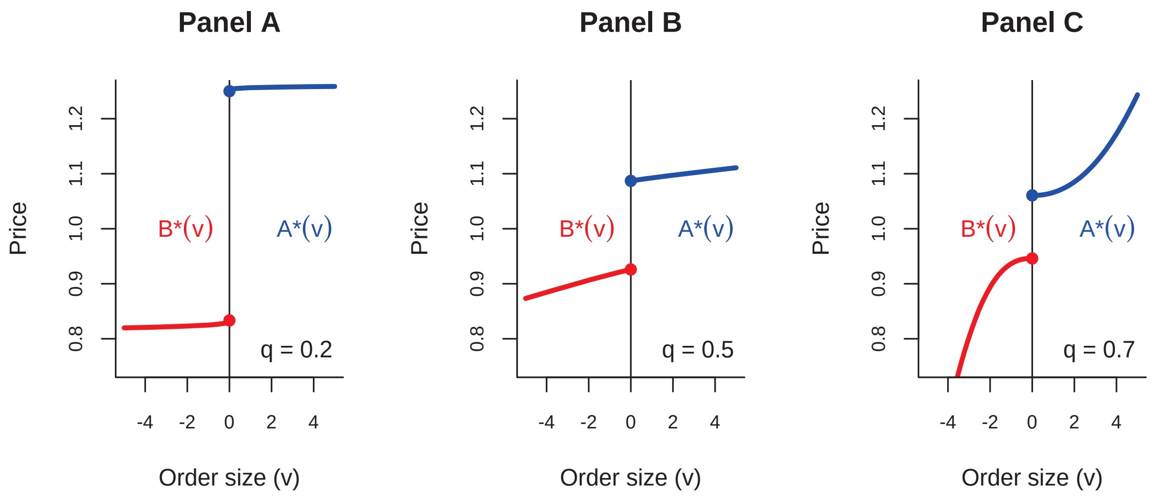

Figure 2: Ask and Bid equilibrium prices for different shares (q) of uninformed traders.

large orders), while the bid-ask spread at zero reduces. This is due to the fact that, when q is high, informed trades happen less often, hence the price process experiences bigger deviations from the fundamental value. Hence, the market maker's potential losses from executing a large order are substantially larger when q is large.

Note that, in a real world market, the arbitrary constants α and β would be uniquely identified by the tick size. Note also that Z_t^M is always positive, and represents the market maker's valuation of owning the stock conditional on all the information available before the last trade and the fact that a trade is occurring at time t. The next remark defines the updating mechanism for the market maker's valuation of the asset $\tilde{z}_i^M = Z_{\tau_i}^M$.

Remark 5 (Update of Market Maker's estimation of the asset value) *Note that if Assumptions A1-A5, as well as Conditions C2-C5, are satisfied, the same steps used in proving Proposition 4 can be used to show that* $Z_t^M = \sum_{i=0}^{\infty} \mathbf{1}_{\{i=L_{t-}\}} \tilde{z}_i^M$ *with*

$$\tilde{z}_i^M = (1-q)\,\tilde{z}_i + q\tilde{z}_{i-1}^M. \tag{15}$$

The above equation states that, in updating her valuation, the market maker will assign a weight q (the probability of the last trader being uninformed) to her previous valuation, and weight $1-q$ (the probability of the trader being informed) to the last trader's valuation.

We can now establish the equilibrium result in the following Theorem

Theorem 6 *Suppose Assumptions A1-A5 are satisfied. For strictly positive constants α and β, there is a unique market equilibrium, $A_t^*(v)$, $B_t^*(v)$, v_i^*, where $A_t^*(v)$ and $B_t^*(v)$ are given,* *respectively, by equations (13) and (14), and*

$$v_i^* = \begin{cases} \left[\frac{1-q}{\alpha(1-\delta)}\left(\frac{q-\delta}{q}\frac{z_i}{z_i^M}-1\right)\right]^{\frac{1-q}{q-\delta}} & \textit{if } \frac{q}{q-\delta}z_i^M < z_i, \\ -\left[\frac{1-q}{\beta(1+\delta)}\left(1-\frac{q+\delta}{q}\frac{z_i}{z_i^M}\right)\right]^{\frac{1-q}{q+\delta}} & \textit{if } z_i < \frac{q}{q+\delta}z_i^M, \\ 0 & \textit{if } \frac{q}{q+\delta}z_i^M \le z_i \le \frac{q}{q-\delta}z_i^M \end{cases}$$

where $z_i^M := Z_{\theta_i}^M$

Proof of Theorem 6. Due to Proposition (4) we know that, for strictly positive constants α and β, equilibrium ask and bid functions are unique and given by equations (13) and (14). Using these expressions for $A_t^*(v)$ and $B_t^*(v)$ in the optimality conditions in Lemma (1) and solving for v completes the proof. ∎

Note that the above equilibrium solution for the order size, v^*, when $\delta = 0$, implies that traders will buy (sell) the asset if and only if their valuation, z_i, is larger (smaller) than the market maker's valuation, z_i^M. When instead the $\delta > 0$, the difference in valuation necessary for a trade to occur needs to be larger in order to account for the trading cost δ. Therefore, in the presence of trading costs, there is an interval of inaction in which no trade occurs even if the valuations of the market maker and the trader differ. This implies that in equilibrium, with $\delta > 0$, number of arrivals and number of trades will be different with strictly positive probability. Nevertheless, one can show that this probability is strictly less than one. Thus, the market maker's (non falsifiable) belief, that number of trades and number of arrivals are the same, is *not* irrational.

3.2 The high frequency (tick-by-tick) equilibrium price process

Given the above characterisations of equilibrium ask and bid pricing functions and the equilibrium trading strategies, we can now characterise the equilibrium price process.

Recall form equation (2) that, since the i-th trade has to occur either at the ask or at the bid price, the price will be

$$\tilde{p}_i = A_{\tau_i}\left(\tilde{v}_i^+\right)\mathbf{1}_{\{\tilde{v}_i>0\}} + B_{\tau_i}\left(\tilde{v}_i^-\right)\mathbf{1}_{\{\tilde{v}_i<0\}}$$

and given the zero utility gain conditions for the market maker (9) and (10) this is

$$\tilde{p}_i = \tilde{z}_i^M\left[\frac{\mathbf{1}_{\{\tilde{v}_i>0\}}}{(1-\delta)} + \frac{\mathbf{1}_{\{\tilde{v}_i<0\}}}{(1+\delta)}\right]. \tag{16}$$

Since, by normalisation, trades start at times after time zero, we need to define the time zero price – that is the price of the asset before any trade as happened. Since the form of the log profit process is common knowledge, we normalise $\tilde{p}_0$ to be equal to the expected value, for any agent, of holding the asset at time zero. That is $\tilde{p}_0 = e^{D_0+\left(\mu+\frac{1}{2}\sigma^2\right)T}$.

From the solutions for the equilibrium ask and bid (13) and (14) we know that

$$\tilde{p}_i = \tilde{z}_{i-1}^M \left[\frac{q}{q-\delta}\left(1+\alpha\tilde{v}_i^{\frac{q-\delta}{1-q}}\right)\mathbf{1}_{\{\tilde{v}_i>0\}} + \frac{q}{q+\delta}\left(1-\beta\left|\tilde{v}_i\right|^{\frac{q+\delta}{1-q}}\right)\mathbf{1}_{\{\tilde{v}_i<0\}}\right].$$

Putting together the last two expressions we have

$$\tilde{p}_i = \tilde{p}_{i-1}c_{1,i}c_{2,i-1}\left(1+\xi_i\left|\tilde{v}_i\right|^{\gamma_i}\right) \tag{17}$$

where

$$\begin{aligned}
c_{1,i} &= \begin{cases} \frac{q}{q-\delta} & \text{if the } i\text{-th trade occurs at ask} \\ \frac{q}{q+\delta} & \text{if the } i\text{-th trade occurs at bid} \end{cases} \\
c_{2,i} &= \begin{cases} 1-\delta & \text{if the } i\text{-th trade occurs at ask and } i>0 \\ 1+\delta & \text{if the } i\text{-th trade occurs at bid and } i>0 \\ 1 & \text{if } i=0 \end{cases} \\
\gamma_i &= \begin{cases} \frac{q-\delta}{1-q} & \text{if the } i\text{-th trade occurs at ask} \\ \frac{q+\delta}{1-q} & \text{if the } i\text{-th trade occurs at bid} \end{cases} \\
\xi_i &= \begin{cases} \alpha & \text{if the } i\text{-th trade occurs at ask} \\ -\beta & \text{if the } i\text{-th trade occurs at bid} \end{cases}
\end{aligned} \tag{18}$$

The above implies that at very high frequency the log price process should be autocorrelated. Moreover, using the relation between order size and cumulated trading volume (1) we have

$$\log\frac{P_{t+s}}{P_t} = \sum_{i=L_t}^{L_{t+s}}\left\{\log\left(1+\xi_i\left|V_{\tau_i}-V_{\tau_{i-1}}\right|^{\gamma_i}\right)+\log c_{1,i}+\log c_{2,i-1}\right\}. \tag{19}$$

That is, there is a direct relationship between price changes and changes in the volume of trade. In particular, the above equation implies that, at high frequency, the volatility of log returns is *i*) stochastic, and *ii*) a function of trade volume $\left|V_{\tau_i}-V_{\tau_{i-1}}\right|$. Moreover, the relationship in equation (19), discussed in detail in Section 4.2, is consistent with a large body of empirical evidence on the joint behaviour of volume, prices, and volatility.[14]

The characterisations of the high frequency price process provided in equations (17) and (19) are a function of *endogenous* variables – respectively of order size, and volume and number of trades. In the next Lemma we characterise the price process as a function of the *exogenous* fundamental value of the asset.

[14]See e.g. Gallant, Rossi, and Tauchen (1992), Farmer and Lillo (2004), Farmer, Lillo, and Mantegna (2003), Potters and Bouchaud (2003).

Lemma 2 (price process and trading times as a function of fundamentals) *Suppose that Assumptions A1-A5 are satisfied and that the market is at the equilibrium. We can define the price process and the time of trades as a function of the exogenous process Z as follows.*

First, normalise τ_0 and $\tilde{p}_0$ as follows

$$\tau_0 = 0,\ P_0 = \tilde{p}_0 = e^{D_0 + \left(\mu + \frac{1}{2}\sigma^2\right)T},\ c_{2,0} = 1.$$

Second, define recursively the trading times

$$\tau_i = \inf \left\{ \theta_j > \tau_{i-1} : \log z_j - \log \tilde{p}_{i-1} \notin \left(b\left(c_{2,i-1}\right), a\left(c_{2,i-1}\right)\right)\right\}, \tag{20}$$

where

$$a(x) = \log\left(\frac{qx}{q-\delta}\right),\ b(x) = \log\left(\frac{qx}{q+\delta}\right), \tag{21}$$

and prices are given by

$$\tilde{p}_i = \frac{1}{c_{2,i}}\left[(1-q)\, z_i + q\tilde{p}_{i-1}c_{2,i-1}\right], \tag{22}$$

where $c_{2,i}$ in equation (18) can be redefined as

$$c_{2,i} = \begin{cases} 1-\delta & \text{if } \log \tilde{z}_i - \log \tilde{p}_{i-1} > a\left(c_{2,i-1}\right) \text{ and } i > 0 \\ 1+\delta & \text{if } \log \tilde{z}_i - \log \tilde{p}_{i-1} < b\left(c_{2,i-1}\right) \text{ and } i > 0 \end{cases} \tag{23}$$

Proof of Lemma 2. Setting $\tau_0 = 0$ is an innocuous normalisation of the time scale. The definition of the equilibrium τ_i in equation (20), as well as $c_{2,i}$ in equation (23), follow from: the agent's optimality conditions in Lemma 1; the form of the equilibrium bid and ask function in Proposition 4; and equation (16), that allows us to replace the market maker's valuation, $\tilde{z}_i^M$, with the price, $\tilde{p}_i$. The definition of the equilibrium price process, $\tilde{p}_i$, in equation (22) follows from the market maker's valuation update in Remark 5 and equation (16). ■

In a nutshell, the above Lemma follows from the observation that, in equilibrium, the trade will occur at the ask price if and only if the valuation of the agent is sufficiently higher than the last recorded market price ($\log z_j - \log \tilde{p}_{i-1} > a\left(c_{2,i-1}\right)$), and at the bid price if instead the agent's valuation is sufficiently lower than the last recorded price ($\log z_j - \log \tilde{p}_{i-1} < b\left(c_{2,i-1}\right)$). This inter-temporal link with the lagged price is due to the fact that the current price is just a linear function of the current market maker's valuation, and this valuation is updated recursively (see Remark 5) due to the presence of uninformed agents. Note that if there were no trading costs we would have $a(.) = b(.) = 0$, implying that agents would always decide to trade either at the ask or bid price.

Since in the above Lemma we have defined the equilibrium trading times and prices as a function on the log valuation ($\log z$), we now turn to the identification of the distribution of this quantity. This is a necessary step to be able to characterise the equilibrium volatility process. In particular, we will look at the distribution of $\log z_i$ conditional on the information set $\mathcal{H}_{i-1} \vee \sigma\{\theta_i\}$ – the information set that contains all the past history of prices, volume of trades, arrivals, and the time of the next arrival. For this task it is convenient to define D_t^{tr} as the value of the log profit that could be inferred observing the valuation of the last agent that arrived on the market. That is

$$d_i^{tr} = \begin{cases} \log z_i - \left(\mu + \frac{\sigma^2}{2}\right)(T - \theta_i) & \forall i \geq 1 \\ D_0 & i = 0 \end{cases}, \quad D_t^{tr} = \sum_{i=0}^{\infty} \mathbf{1}_{\{i=N_t\}} d_i^{tr}. \tag{24}$$

Note that the value of $D_{\tau_i}^{tr}$ can be always inferred from the last occurred trade due to the fact that agents preferences are common knowledge. The distribution of d_i^{tr} is characterised in the following lemma.

Lemma 3 *Suppose that Assumptions A1-A5 are satisfied. Then*

$$\mathbb{P}\left[d_i^{tr} \leq x | \mathcal{H}_{i-1}, \theta_i\right] = \mathbb{P}\left[D_{\theta_i} \leq x | \mathcal{H}_{i-1}, \theta_i\right]$$

$$= (1-q)\sum_{j=1}^{i-1} q^{i-1-j}\mathbb{P}\left[d_j^{tr} + \varepsilon_{i,j} \leq x | d_j^{tr}, \Delta_{i,j}\right] + q^{i-1}\mathbb{P}\left[d_0^{tr} + \varepsilon_{i,0} \leq x | d_0^{tr}, \Delta_{i,0}\right]$$

where $\Delta_{i,j} := \theta_i - \theta_j$, $\varepsilon_{i,j} := \mu\Delta_{i,j} + \sigma\sqrt{\Delta_{i,j}}\eta_{i,j}$, *and* $\eta_{i,j} \sim N(0,1)$ *is independent of* d_j^{tr} *and* $\Delta_{i,j}$ *for all* $j < i$.

The proof of the above Lemma is quite involved, and we therefore report it in Appendix A.1. The rationale behind it is nevertheless quite intuitive. At each point in time either an informed (with probability $1-q$) or an uniformed (with probability q) agent arrives to the market and, from equation (24), her d_i^{tr} is simply a (log) linear function of her expected payoff (z_i) from holding the asset. Recall that $\mathcal{H}_{i-1}$ contains all the past history of arrivals and volume of trade, and based on this information and the knowledge of the time of the last arrival (θ_i) only, informed and uninformed agents are indistinguishable. This implies that $\mathbb{P}[d_i^{tr} \leq x|\mathcal{H}_{i-1}, \theta_i, I_i] = \mathbb{P}[d_i^{tr} \leq x|\mathcal{H}_{i-1}, \theta_i, U_i]$. Moreover, only the arrival of an informed agent can add new relevant information about the fundamental. Therefore, the last relevant information is revealed by the last informed arrival, and the probability of this being the j-th arrival is simply $(1-q)q^{i-1-j}$. Moreover, if no informed agent ever arrived to the market before the i-th arrival, the only relevant information is the common knowledge d_0^{tr}, and this event might occur with probability q^{i-1}. Furthermore, since the innovations in D are simple independent Brownian motion differences, the ε terms appear.

Note that since prices are uniquely determined by D^{tr} (through Lemma (2) and equation (24)), Lemma 3 also characterises the distribution of prices. Therefore, if the D^{tr} process were to converge in distribution, this would also imply (by continuos mapping theorem), the convergence in distribution of the price process. This limiting distribution is the focus of the next sub section.

3.3 The medium frequency equilibrium price process

Having characterised the price process and the distribution of agents' valuations on the tick-by-tick time scale, we now turn to the analysis of the equilibrium price process at lower frequencies. This is needed in order to establish the link between information based trading and, as an equilibrium outcome, endogenous stochastic volatility.

In this section we make one simplifying assumption regarding the arrival process: we consider a Poisson process. The assumption of a Poisson arrival process is not strictly necessary, since all that we need to derive our results is that the arrival process satisfies a set of properties (described in detail below) that hold almost surely for a Poisson process. In order to simplify exposition, we consider a process with constant intensity but we could as well handle a process with time varying intensity. Considering a fixed intensity arrival process has also the advantage that the only channel through which stochastic volatility will arise is the information based trading. Moreover, assuming constant intensity is a very minor restriction since, as we will show, the equilibrium medium and low frequency price and volatility processes, as well as the number of trades process, will be independent of the arrival process.

By medium frequency we mean a time interval in which the number of arrivals is vey large. To model this mathematically, we send the intensity of arrivals to infinity. This modelling approach has the advantage that, as the intensity of arrivals goes to infinity, the constraint to trade due to the exogenous arrival times will disappear.

The key result established in this section is summarised in the following Theorem.

Theorem 7 (Limiting Price Process) *Suppose the process D is given by*

$$dD_t = \mu dt + \sigma dW_t^d, \qquad D_0 = const$$

with W^d being a standard Brownian motion with respect to $(\mathcal{F}_s)_{s\geq 0}$. Suppose also that Λ is a Poisson process, with intensity parameter λ, defined on $[0,+\infty)$, and $\mathcal{F}_\infty^\Lambda$ is independent of $\mathcal{F}_\infty^W$. Then there exists a sequence of Poisson arrival processes N^n, satisfying $\mathbb{P}[N_t^n = \Lambda_{tn},\ t\in[0,T]] = 1$, such that the equilibrium price process P^n resulting from any sequence of markets $\mathcal{M}^n(N^n, D, S^n, U^n)$ satisfying Assumptions A2-A6, weakly converges *in Skorokhod topology. The limit price process P is a functional of W, a standard Brownian Motion, independent of the number of arrivals process, Λ. Moreover, this functional has the form*

$$P_t \overset{d}{=} \prod_{i=1}^{L_t^p} \phi_i \left(\frac{q}{\phi_{i-1}} + 1 - q \right) \tag{25}$$

where $\phi_0 = 1$, $\tau_0 = 0$ and, for any $i \geq 1$

$$\phi_i := \begin{cases} \frac{q}{q-\delta} & \text{if } \sigma \left(W_{\tau_i} - W_{\tau_{i-1}} \right) - \frac{\sigma^2}{2} (\tau_i - \tau_{i-1}) = a \left(\frac{q}{\phi_{i-1}} + 1 - q \right) \\ \frac{q}{q+\delta} & \text{if } \sigma \left(W_{\tau_i} - W_{\tau_{i-1}} \right) - \frac{\sigma^2}{2} (\tau_i - \tau_{i-1}) = b \left(\frac{q}{\phi_{i-1}} + 1 - q \right) \end{cases}.$$

$$\tau_i = \inf \left\{ t \geq \tau_{i-1} : \sigma \left(W_t - W_{\tau_{i-1}} \right) - \frac{\sigma^2}{2} (t - \tau_{i-1}) \notin \left[b \left(\frac{q}{\phi_{i-1}} + 1 - q \right), a \left(\frac{q}{\phi_{i-1}} + 1 - q \right) \right] \right\}$$

$$L_t^p = \sum_{j=1}^{\infty} \mathbf{1}_{\{\tau_j \leq t\}}$$

and $a(.)$ and $b(.)$ are defined in equation (21).

The rest of this section is dedicated to prove the above theorem. But before undertaking this task, we can use the above result to characterise the first two moments of the price process in the following corollary.

Corollary 1 (Volatility of the Limiting Price Process) *The distribution of ϕ_i is, for $i > 1$*

$$\phi_i := \begin{cases} q/(q-\delta) & w.p.\ \mathbf{1}_{\left\{\phi_{i-1} = \frac{q}{q-\delta}\right\}} \frac{(q-\delta)(1+q)}{2q(1-\delta)} + \mathbf{1}_{\left\{\phi_{i-1} = \frac{q}{q+\delta}\right\}} \frac{(q-\delta)(1-q)}{2q(1+\delta)} \\ q/(q+\delta) & w.p.\ \mathbf{1}_{\left\{\phi_{i-1} = \frac{q}{q-\delta}\right\}} \frac{(q+\delta)(1-q)}{2q(1-\delta)} + \mathbf{1}_{\left\{\phi_{i-1} = \frac{q}{q+\delta}\right\}} \frac{(q+\delta)(1+q)}{2q(1+\delta)} \end{cases}.$$

and for $i = 1$

$$\phi_1 := \begin{cases} q/(q-\delta) & w.p.\ \frac{q-\delta}{2q} \\ q/(q+\delta) & w.p.\ \frac{q+\delta}{2q} \end{cases}.$$

Implying the conditional moments for $i > 1$

$$\mathbb{E}\left[\frac{\tilde{p}_i}{\tilde{p}_{i-1}} | \mathcal{F}_{\tau_{i-1}}^W \right] = 1,\ \ Var\left(\frac{\tilde{p}_i}{\tilde{p}_{i-1}} | \mathcal{F}_{\tau_{i-1}}^W \right) = \frac{\delta^2(1-q^2)}{q^2-\delta^2}.$$

Proof. The proof is reported in Appendix A.1. ■

Remark 8 (Ergodic Distribution) *The two state Markov process for ϕ_i defined above has the following ergodic distribution:*

$$\phi_i := \begin{cases} q/(q-\delta) & w.p.\ \frac{(q-\delta)(1-\delta)}{2(q+\delta^2)} \\ q/(q+\delta) & w.p.\ \frac{(q+\delta)(1+\delta)}{2(q+\delta^2)} \end{cases}.$$

The above corollary makes clear that, on the trade time scale, the price process, is characterised by constant volatility. Since trade and calendar time differ (due to the inaction region generated by the bid ask spread), this implies that the volatility on the calendar time scale will be driven by the number of trades process and, since this process is stochastic, the calendar time volatility itself is stochastic. The properties of the trade by trade volatility characterised above are discussed in detail in section 4.2.

To prove Theorem 7 we first construct a new process (Y) that has exactly the same distribution as the shadow valuation of the asset (D^{tr}) in Lemma 3. Then, we establish weak convergence of this process in Skorokhod topology. And finally, to complete the proof via the Continuous Mapping Theorem, we establish the continuity of the mapping between the shadow valuation process and trade prices.

In order to define a sequence of markets as in the above Theorem, we need to define the processes N^n, S^n, and U^n. Given these processes, the price process P^n is obtained from Theorem 6 and Equation 17.

First, we define the process of traders' arrival N^n. Consider any given Poisson process Λ, with intensity λ, and corresponding arrival times $\gamma_i := \inf \{t \geq 0 : \Lambda_t \geq i\}$ that are independent of $\mathcal{F}^W_\infty$. The arrival intensity of the n-th market is constructed as $n\lambda$. For any of these Λ processes, we introduce regularity conditions by considering the following sets:

$$\Omega_1 = \left\{ \omega \in \Omega : \lim_{i \to +\infty} \frac{\sum_{j=1}^{\lfloor xi \rfloor} (\gamma_j - \gamma_{j-1})^2}{\sum_{j=1}^{i} (\gamma_j - \gamma_{j-1})^2} = x \text{ for any } x \in [0,1] \right\},$$

$$\Omega_2 = \left\{ \omega \in \Omega : \max_{i \leq k} (\gamma_i - \gamma_{i-1}) < \infty \text{ for all } k \in \mathbb{N}_+ \right\},$$

$$\Omega_3 = \cup_{k=1}^{\infty} \cap_{i=k}^{\infty} \{\omega \in \Omega : (\gamma_i - \gamma_{i-1}) \leq 2 \log (i)\},$$

$$\Omega_4 = \left\{ \omega \in \Omega : \lim_{n \to \infty} \frac{\Lambda_{tn}}{n} = t\lambda \text{ for any } t \in [0,T] \right\},$$

$$\Omega_5 = \left\{ \omega \in \Omega : \lim_{n \to \infty} \frac{\sum_{i=1}^{\Lambda_{tn}} (\bar{\gamma}_i - \bar{\gamma}_{i-1})}{\Lambda_{tn}} = \lambda^{-1} \text{ for any } t \in [0,T] \right\},$$

where the operator $\lfloor \cdot \rfloor$ returns the largest integer smaller than its argument. Note that the above regularity conditions are satisfied by the Poisson process almost surely since: *a*) from the strong Law of Large Numbers $\mathbb{P}(\Omega_i) = 1$ for $i = 1, 4$; *b*) $\mathbb{P}(\Omega_3) = 1$ from the Borel-Cantelli Lemma; *c*) $\mathbb{P}(\Omega_2) = 1$ is a property of the Poisson process; *d*) condition Ω_5 is simply a strong law of large number requirement; *e*) $\mathbb{P}(\Omega_5) = 1$ for a Poisson process. Nevertheless, the fact that these regularity conditions are satisfied almost surely does not guarantee that they will be satisfied for every $\omega \in \Omega$, since on some zero probability sets they could be violated. Therefore, since we will be conditioning on paths of Λ, we need to modify the Λ

process on the zero probability sets to ensure that these properties will hold for every $\omega \in \Omega$.

The modification of the Poisson process Λ, denoted $\bar{\Lambda}$, that satisfies the above regularity conditions for each $\omega \in \Omega$, is given by[15]

$$\bar{\gamma}_0(\omega) = 0, \ \ \bar{\gamma}_i(\omega) = \begin{cases} \gamma_i(\omega) & \text{if } \omega \in \cap_{j=1}^4 \Omega_j \\ \bar{\gamma}_{i-1}(\omega) + \frac{1}{\lambda} & \text{if } \omega \in \Omega \backslash \left(\cap_{j=1}^4 \Omega_j\right) \end{cases}, \ \ \bar{\Lambda}_t = \sum_{i=1}^{\infty} \mathbf{1}_{\{\bar{\gamma}_i \leq t\}}.$$

The corresponding sequence of traders arrival processes can now be defined as

$$N_t^n = \bar{\Lambda}_{tn}, \tag{26}$$

$$\theta_i^n = \frac{\bar{\gamma}_i}{n}. \tag{27}$$

Note that the intensity of the counting process N^n is simply λn. So, as $n \to \infty$, the intensity of arrivals goes to infinity. Moreover, note that the arrival process just defined satisfies Assumption A1.

Second, the above Theorem requires us to consider a sequence of markets. Nevertheless, we have established that, in equilibrium, market prices are uniquely determined by D^{tr} through Lemma (2) and equation (24), and we have already characterised in Lemma 3 the distribution of D^{tr}. As a consequence, since we are aiming to prove only weak convergence of the price process, it is enough to construct a sequence of processes, Y^n, that have the same distribution as the D^{tr} process that results from the market equilibrium.

In the following Lemma we construct the Y^n process such that $\mathcal{L}\left(D^{tr,n} | \mathcal{F}_\infty^{\bar{\Lambda}}\right) = \mathcal{L}\left(Y^n | \mathcal{F}_\infty^{\bar{\Lambda}}\right)$, where $\mathcal{L}\left(\cdot | \mathcal{F}_\infty^{\bar{\Lambda}}\right)$ denotes the finite dimensional distribution and $D^{tr,n}$ denotes the D^{tr} process in the market $\mathcal{M}^n$. Therefore, for any fixed n, Y_t^n has the same information content as the value of the log profit that could be inferred observing the valuation of the last agent that arrived (before time t) on the market $\mathcal{M}^n$. That is, the process Y^n can be thought of as a value process of the log profit at arrival times.

Lemma 4 *Fix a process N^n given by Equation (26), and any market $\mathcal{M}^n(N^n, D, S^n, U^n)$ satisfying Assumptions A1-A6. Let $D^{tr,n}$ be the resulting value of the log profit from the agents' point of view, given in equation (24), and that uniquely determines the equilibrium price process (through Lemma (2) and equation (24)).*

Consider the process Y^n, on the interval $[0, T]$, given by

$$Y_t^n = \sum_{j=0}^{\infty} \mathbf{1}_{\{N_t^n = j\}} y_j^n, \tag{28}$$

$$y_0^n = D_0, \tag{29}$$

[15]Note that $\bar{\Lambda}$ is an adapted process since the filtration we use satisfies the usual conditions.

$$y_i^n = \sum_{j=0}^{i-1} \zeta_{i-1,j} \left(y_j^n + \varepsilon_{i,j}^n \right), \tag{30}$$

where $\varepsilon_{i,j}^n := \mu\Delta_{i,j}^n + \sigma\sqrt{\Delta_{i,j}^n}\eta_{i,j}$, $\Delta_{i,j}^n := \theta_i^n - \theta_j^n$, $\eta_{i,j}$ *is an independent standard Gaussian, and* $\zeta_{i-1,j}$ *is a* 0 *or* 1 *random variable such that:* $\sum_{j=0}^{i-1} \zeta_{i-1,j} = 1$*;*

$$\mathbb{P}(\zeta_{i-1,j} = 1) = \begin{cases} (1-q)\, q^{i-1-j} & \text{for } j > 0 \\ q^{i-1} & \text{for } j = 0 \end{cases}; \tag{31}$$

and $\sigma\left\{\zeta_{i-1,j}\right\}_{j=0}^{i-1} \perp \vee_{i' \neq i} \sigma\left\{\zeta_{i'-1,j}\right\}_{j=0}^{i'}$*;* $\vee_i \sigma\left\{\zeta_{i-1,j}\right\}_{j=0}^{i-1} \perp \mathcal{F}_\infty^{\bar{\Lambda}}$*. Then we have*

$$\mathcal{L}\left(Y^n | \mathcal{F}_\infty^{\bar{\Lambda}}\right) = \mathcal{L}\left(Y^n | \mathcal{F}_T^{N^n}\right) = \mathcal{L}\left(D^{tr,n} | \mathcal{F}_T^{N^n}\right) = \mathcal{L}\left(D^{tr,n} | \mathcal{F}_\infty^{\bar{\Lambda}}\right) \tag{32}$$

$$\mathcal{L}\left(\mathcal{P}Y^n | \mathcal{F}_\infty^{\bar{\Lambda}}\right) = \mathcal{L}\left(\mathcal{P}Y^n | \mathcal{F}_T^{N^n}\right) = \mathcal{L}\left(P^n | \mathcal{F}_T^{N^n}\right) = \mathcal{L}\left(P^n | \mathcal{F}_\infty^{\bar{\Lambda}}\right) \tag{33}$$

where $\mathcal{P}$ *is the mapping from the value process of log profits at arrival times to equilibrium prices (defined in Lemma 2 and equation (24)), and* P^n *is the equilibrium price process of the* $\mathcal{M}^n$ *market.*

Proof. Equation (32) follows from direct comparison of the distribution of Y^n and the one of D^{tr} in Lemma 3 and the fact that Y^n and D^{tr} are defined on $[0, T]$. Equation (33) follows from the fact that equations (2) and (24) identify a unique mapping between D^{tr} and the equilibrium price process, and the fact that prices are defined on $[0, T]$. ■

The above Lemma makes clear that, to establish and characterise the convergence of the Equilibrium price process, it is enough to establish and characterise the convergence of the Law of Y^n and the continuity of the mapping $\mathcal{P}$.

For convenience and clarity of exposition (and to avoid some technical issues arising from zero probability sets) we define a new (random) probability measure $\bar{\mathbb{P}}$ to remove the conditioning in equations (32) and (33). That is, let $\bar{\mathbb{P}}$ be a measure on $\mathcal{F}_\infty^{\bar{\Lambda}} \vee_n \mathcal{F}_T^{Y^n}$, given by the regular version of the kernel $\mathbb{P}\left(G | \mathcal{F}_\infty^{\bar{\Lambda}}\right)$, i.e. for any $G \subset \mathcal{F}_\infty^{\bar{\Lambda}} \vee_n \mathcal{F}_T^{Y^n}$ we have that $\bar{\mathbb{P}}(G) = \mathbb{P}\left(G | \mathcal{F}_\infty^{\bar{\Lambda}}\right)$. Such a $\bar{\mathbb{P}}$ measure exist and is unique due to Theorem 6.4 of Kallenberg (2002). Therefore, convergence under $\bar{\mathbb{P}}$ (i.e. $\bar{\mathcal{L}}(Y^n) \to \bar{\mathcal{L}}(Y)$) implies convergence under the original $\mathbb{P}$ measure (i.e. $\mathcal{L}(Y^n) \to \mathcal{L}(Y)$).

Using the definition of Y^n (in Lemma 4) and $\bar{\mathbb{P}}$, we can establish the first convergence result needed to prove Theorem 7.

Proposition 9 *Consider* $\bar{Y}_t^n := \sum_{i=0}^{\infty} \mathbf{1}_{\{N_t^n = i\}} \left[y_i^n + \mu (T - \theta_i^n) \right]$*. Then the sequence of processes* $\left(\bar{Y}^n, e^{\bar{Y}^n}, \bar{\theta}^n, \bar{\mathcal{F}}^n\right)$*, where* $\bar{\mathcal{F}}_t^n := \mathcal{F}_\infty^{\bar{\Lambda}} \vee \mathcal{F}_t^{\bar{Y}^n}$ *and* $\bar{\theta}_t^n := \theta_{N_t^n}^n$*, weakly converges in Skorokhod topology on* $\mathbb{D}([0, T])$ *(the space of cádlág processes in the* $[0, T]$ *interval), as* $n \to \infty$*,* *to* $\left(\bar{Y}, e^{\bar{Y}}, \bar{\theta}, \bar{\mathcal{F}}\right)$*, where* $\bar{\mathcal{F}}_t := \mathcal{F}_\infty^{\bar{\Lambda}} \vee \mathcal{F}_t^{\bar{Y}}$*,* $\bar{\theta}_t = t$*, and* $\bar{Y}_t = \sigma W_t$ *where* W *is a standard Brownian motion on its own augmented filtration and it is independent of* $\mathcal{F}_\infty^{\bar{\Lambda}}$*.*

The proof of the above proposition is quite technical, and requires establishing some intermediate results, and is therefore reported in Appendix A.2. Nevertheless, its core is quite simple to grasp. The $\bar{Y}^n$ process is, by construction, a long memory process. Therefore, to establish the above limiting results, we show that its serial correlation decays at a fast enough rate to ensure mixingale convergence. With this result at hand, we then prove that $\bar{Y}$ is proportional to a standard Brownian motion by showing that it is a local martingale with quadratic variation equal to $\sigma^2 t$.

Since, by definition, $Y_t^n \equiv \bar{Y}_t^n + \mu\left(\bar{\theta}_t^n - T\right)$, and the above Proposition states the joint convergence of $\bar{Y}^n$ and $\bar{\theta}^n$, we have that a similar convergence result holds for Y^n.

Corollary 2 *The sequence of processes* $\left(Y^n, e^{Y^n}, \bar{\theta}^n, \bar{\mathcal{F}}^n\right)$*, weakly converges in Skorokhod topology on* $\mathbb{D}([0,T])$*, as* $n \to \infty$*, to* $\left(Y, e^Y, \bar{\theta}, \bar{\mathcal{F}}\right)$*, and* $Y_t = \mu(t-T) + \sigma W_t$ *where* W *is a standard Brownian motion on its own augmented filtration and it is independent of* $\mathcal{F}_\infty^{\bar{\Lambda}}$

Given the above convergence result for Y^n, and since (from Lemma 4) $\mathcal{L}\left(\mathcal{P}Y^n | \mathcal{F}_\infty^{\bar{\Lambda}}\right) = \mathcal{L}\left(P^n | \mathcal{F}_\infty^{\bar{\Lambda}}\right)$, all we need to complete the proof of Theorem 7 is to establish that the sequence of processes $\mathcal{P}Y^n$ converges – that is, we need to establish the convergence of the sequence of equilibrium price processes (P^n). We do so by *i*) breaking the map $\mathcal{P}$ into two maps, $\mathcal{P}_1$ and $\mathcal{P}_2$, and establishing *ii*) the convergence of the processes $\mathcal{P}_1 Y^n$ and *iii*) the continuity of the map $\mathcal{P}_2$.

First, the map $\mathcal{P}_1 : \mathbb{D}([0,T]) \to \mathbb{D}([0,T])$ is given by

$$(\mathcal{P}_1 f)(t) := f(t) + \left(\mu + \frac{\sigma^2}{2}\right)(T - \sup\{s \le t : f(s_-) \neq f(t)\}), \quad \forall f \in \mathbb{D}[0,T].$$

Note that $\mathcal{P}_1$ identifies the arrival times. In particular, the *sup* component returns the previous period arrival time, when f is a path of (the piecewise constant) process Y^n, and it is equal to t if f is a path of the (limiting) continuous process Y. Thus we have

$$(\mathcal{P}_1 Y^n)_t = Y_t^n + \left(\mu + \frac{\sigma^2}{2}\right)\left(T - \bar{\theta}_t^n\right) =: H_t^n. \tag{34}$$

where H is the valuation of the agent that last arrived on the market (note that $\mathcal{P}_1 Y_t^n$ is just the log of the expectation of $e^{Y_T^n}$).

It follows from Corollary 2 (and Corollary VI.3.33.b of Jacod and Shiryaev (2003)) that $\left(H^n, \bar{\theta}^n, \bar{\mathcal{F}}^n\right)$, weakly converges in Skorokhod topology on $\mathbb{D}([0,T])$, as $n \to \infty$, to $\left(H, \bar{\theta}, \bar{\mathcal{F}}\right)$, and

$$H_t = \frac{\sigma^2}{2}(T-t) + \sigma W_t, \; t \in [0,T] \tag{35}$$

where W is a standard Brownian motion on its own augmented filtration.

Second, note that the price process can be recovered as $P^n \equiv \mathcal{P}_2\mathcal{P}_1Y^n$, where $\mathcal{P}_2 : \mathbb{D}([0,T]) \to \mathbb{D}([0,T])$ is defined by $(\mathcal{P}_2 f)(t) := g(\tau^f_{L^f_t})$ for any $f \in \mathbb{D}[0,T]$, where $L^f_t := \sum_{i\geq 0} \mathbf{1}_{\{\tau^f_i \leq t\}}$, and $g(\cdot)$ and τ^f are obtained through the following recursion

$$\tau^f_0 = 0,\ g_0 = e^{f(0)},\ c^f_{2,0} = 1,$$

$$\tau^f_i = \inf\left\{t > \tau^f_{i-1} : f(t) - \ln g\left(\tau^f_{i-1}\right) \notin \left(b\left(c^f_{2,i-1}\right), a\left(c^f_{2,i-1}\right)\right)\right\}, \tag{36}$$

where $a(.)$ and $b(.)$ are defined in Equation (21),

$$c^f_{2,i} = \begin{cases} 1-\delta & \text{if } f\left(\tau^f_i\right) - \ln g\left(\tau^f_{i-1}\right) > a\left(c^f_{2,i-1}\right) \text{ and } i > 0 \\ 1+\delta & \text{if } f\left(\tau^f_i\right) - \ln g\left(\tau^f_{i-1}\right) < b\left(c^f_{2,i-1}\right) \text{ and } i > 0 \end{cases} \tag{37}$$

and

$$g\left(\tau^f_i\right) = \frac{1}{c^f_{2,i}}\left[(1-q)\, e^{f\left(\tau^f_i\right)} + qg\left(\tau^f_{i-1}\right) c^f_{2,i-1}\right]. \tag{38}$$

Note that the above recursion is analogous to the one defining the price process and trading times as a function of fundamentals in Lemma 2 where, in particular, the equation for stopping times τ^f_i corresponds to the times of trades in equation (20), and the equation for the update of the function $g(\cdot)$ is nothing but the price evolution defined in equation (22).

Consider the following set of functions $\mathcal{C}$

$$\mathcal{C} := \left\{f \in \mathbb{C}[0,T] : L^f_T < \infty,\ \tau^f_i = \tau^{f+}_i,\ L^f_{T-} = L^f_T,\ \forall i = 1,\ldots,L^f_T,\ K_\tau > 0,\ \tau^f_1 \neq 0\right\} \tag{39}$$

where

$$\tau^{f+}_i := \inf\left\{t \geq \tau^f_{i-1} : f(t) - \ln g\left(\tau^f_{i-1}\right) \notin \left[b\left(c^f_{2,i-1}\right), a\left(c^f_{2,i-1}\right)\right]\right\}$$

$$K_\tau := \min\left\{\min_{i=1,\ldots,L^f_T}\left(\tau^f_i - \tau^f_{i-1}\right); T - \tau^f_{L^f_T}\right\}$$

that is, the set of continuous functions characterised by spaced apart hitting times, and that cross the boundaries, defined by $a(\cdot)$ and $b(\cdot)$, upon reaching them. Note that when f belong to the set $\mathcal{C}$, we have that $(\mathcal{P}_2 f)(t) = \exp\left\{f\left(\tau^f_{L^f_t}\right)\right\}$ where $L^f_t := \sum_{i\geq 0} \mathbf{1}_{\{\tau^f_i \leq t\}}$, and τ^f, are obtained through the following recursion

$$\tau^f_0 = 0,\ c^f_{2,0} = 1,$$

$$\tau^{f+}_i = \tau^f_i = \inf\left\{t > \tau^f_{i-1} : f(t) - f\left(\tau^f_{i-1}\right) \notin \left(b\left(c^f_{2,i-1}\right), a\left(c^f_{2,i-1}\right)\right)\right\}, \tag{40}$$

where

$$c_{2,i}^{f}=\begin{cases}1-\delta & \text{if } f\left(\tau_i^f\right)-f\left(\tau_{i-1}^f\right)=a\left(c_{2,i-1}^f\right) \text{ and } i>0,\\ 1+\delta & \text{if } f\left(\tau_i^f\right)-f\left(\tau_{i-1}^f\right)=b\left(c_{2,i-1}^f\right) \text{ and } i>0.\end{cases} \tag{41}$$

Note that a path of Brownian motion (with or without a constant drift) belongs to the set $\mathcal{C}$ almost surely (since $\left|b\left(c_{2,i-1}^f\right)\right|$, $a\left(c_{2,i-1}^f\right)>0$ for all i, i.e. since, at any given hitting time, the distance between the current value of the function and the next hitting bound is strictly positive).

To establish the convergence in distribution of the equilibrium price processes (P^n), we need to establish the continuity of the map $\mathcal{P}_2$ (on the set $\mathcal{C}$), which is done in Lemma 5 below (the proof of the Lemma is reported in Appendix A.1).

Lemma 5 *For any function f belonging to the set $\mathcal{C}$ defined in equation (39), the map $\mathcal{P}_2$ defined by equations (36)-(38) is continuous in Skorokhod topology at f.*

With the above result at hand, we can complete the proof of Theorem 7.

Proof of Theorem 7. Observe that

$$\lim_{n\to\infty}\mathcal{L}\left(P^n|\mathcal{F}_\infty^{\bar{\Lambda}}\right)=\lim_{n\to\infty}\mathcal{L}\left(\mathcal{P}Y^n|\mathcal{F}_\infty^{\bar{\Lambda}}\right)=\lim_{n\to\infty}\mathcal{L}\left(\mathcal{P}_2H^n|\mathcal{F}_\infty^{\bar{\Lambda}}\right)=\mathcal{L}\left(\mathcal{P}_2H|\mathcal{F}_\infty^{\bar{\Lambda}}\right)$$

where the first equality is due to equation (33), the second equality is due to the definition of the map $\mathcal{P}_2$, and the last equality is due to the convergence of H^n established in Corollary 2 and the continuity of the map $\mathcal{P}_2$ proved in Lemma 5.

The conclusion of the Theorem follows once we observe that $H\in\mathcal{C}$. Therefore the limiting price process, P, exists and is given by

$$P_t \stackrel{d}{:=} \left(\mathcal{P}_2H\right)(t)=\exp\left\{H_{\tau^H_{L_t^H}}\right\}=\prod_{i=1}^{L_t^H}c_{2,i-1}^H\phi_i, \tag{42}$$

where

$$\phi_i:=\begin{cases}q/\left(q-\delta\right) & \text{if } H_{\tau_i^H}-H_{\tau_{i-1}^H}=a\left(c_{2,i-1}^H\right) \text{ and } i>0\\ q/\left(q+\delta\right) & \text{if } H_{\tau_i^H}-H_{\tau_{i-1}^H}=b\left(c_{2,i-1}^H\right) \text{ and } i>0\end{cases}.$$

The statement of the theorem follows upon observing the form of H in equation (35), and that: $c_{2,i}^H\equiv q/\phi_i+1-q$, $\tau^H\equiv\tau$, $L^p\equiv L^H$. ∎

Note that the medium frequency price process in equation (42) clearly does not depend on the volume of trade, nor on the traders' arrival process, but only on the *number* of trades, L_t^H. Moreover, the above shows that the volatility of the price process is affected by the parameters q and δ.

3.4 The low, and ultra-low, frequency equilibrium price processes

In order to characterise the low frequency price process behaviour, we will send the number of trades between time s and t, that is $L_t^p - L_s^p$, to infinity, and we will study the volatility of the limiting distribution of the (appropriately scaled) log return. This task is complicated by the fact that *a*) the sequence of ϕ's, that drives the price process, is serially dependent (from Corollary 1), and *b*) the time between arrivals is also dependent (from Theorem 7, since this quantity too depends upon the sequence of ϕ's). Note also that, although we have already obtained the limiting trade by trade volatility in Corollary 3.3, on the calendar time scale the volatility of the price process is also affected by the average time between consecutive trades, thus altering the limiting distribution on this time scale.

In what follows, we establish that the (centred) calendar time of trades is a mixingale and that the sample mean of times between consecutive trades (i.e. the inter-arrival time of trades) converges almost surely to a constant. Based on this result, we construct a (novel) central limit type argument to characterise the limiting volatility of the price process on the calendar time scale.

Lemma 6 (Expected Inter-arrival Time of Trades) *Let τ_n denote the stopping time defined in Theorem 7. Then*

$$\frac{\tau_n}{n} \underset{n\to\infty}{\longrightarrow} \mu_\tau := \frac{2}{\sigma^2}\left[\log\frac{q-\delta}{q(1-\delta)} + \frac{(q+\delta)(1+\delta)}{2\,(q+\delta^2)}\log\frac{(1-\delta)(q+\delta)}{(1+\delta)(q-\delta)}\right] \quad a.s.. \tag{43}$$

Moreover, for any $\omega \in \Omega$ such that $\lim_{n\to\infty}\frac{\tau_n(\omega)}{n} = \mu_\tau$, we have

$$\frac{L_t^p}{t}(\omega) \underset{t\to\infty}{\longrightarrow} \frac{1}{\mu_\tau}. \tag{44}$$

The proof of the above Lemma is technical and we therefore confine it to Appendix A.1. Nevertheless, its mechanic is simple since it is based on establishing that the serial dependence of inter-arrival times decays exponentially, therefore the sequence of (centred) inter-arrival times is a ($\mathcal{L}^2$) mixingale. This result allows us to characterise the low frequency distribution of log returns in the following proposition.

Proposition 10 (Low, and Ultra-low, Frequency Return Distributions) *Let μ_τ be as defined in Lemma 6. The asymptotic distributions of log returns are:*

$$\frac{\log\frac{P_t}{P_s} - \frac{\sigma^2}{2}(s-t)}{\sqrt{L_t^p - L_s^p}} \underset{t-s\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(0, \sigma^2\mu_\tau\right), \tag{45}$$

$$\frac{\log\frac{P_t}{P_s} - \frac{\sigma^2}{2}(s-t)}{\sqrt{t-s}} \underset{t-s\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(0, \sigma^2\right). \tag{46}$$

Proof. Define $\tau'_k := \inf\{n \in \mathbb{N} : n \geq \tau_k\}$. Fix an $s \geq 0$ and an $\omega \in \mathcal{C}$ where $\mathcal{C} := \left\{\omega \in \Omega : \lim_{n\to\infty}, \frac{\tau_n(\omega)}{n} = \mu_\tau\right\}$, and observe that

$$\frac{\tau'_{L_t^p}}{t}(\omega) \equiv \left(\frac{\tau'_{L_t^p} - \tau_{L_t^p}}{L_t^p}(\omega) + \frac{\tau_{L_t^p}}{L_t^p}(\omega)\right)\frac{L_t^p}{t}(\omega).$$

By Lemma 6, for any $\omega \in \mathcal{C}$, we have that $\lim_{t\to\infty}\frac{L_t^p}{t}(\omega) = \frac{1}{\mu_\tau}$, implying that

$$0 \leq \lim_{t\to\infty}\frac{\tau'_{L_t^p} - \tau_{L_t^p}}{L_t^p}(\omega) \leq \lim_{t\to\infty}\frac{1}{L_t^p}(\omega) = 0.$$

Hence

$$\left(\frac{\tau'_{L_t^p} - \tau_{L_t^p}}{L_t^p}\frac{L_t^p}{t}\right)(\omega) \underset{t\to\infty}{\longrightarrow} 0.$$

Similarly, from Lemma 6 and the definition of $\mathcal{C}$, we have that for $\omega \in \mathcal{C}$

$$\left(\frac{\tau_{L_t^p}}{L_t^p}\frac{L_t^p}{t}\right)(\omega) \underset{t\to\infty}{\longrightarrow} 1.$$

Moreover, since $\mathbb{P}(\mathcal{C}) = 1$ by Lemma 6, we have $\frac{\tau'_{L_t^p}}{t} \underset{t\to\infty}{\longrightarrow} 1$ a.s., which implies $\frac{\tau'_{L_t^p} - \tau'_{L_t^s}}{t-s} \underset{t-s\to\infty}{\longrightarrow} 1$ a.s.. Thus, it follows from the Anscombe's Theorem (see e.g. Gut (2009), Theorem 1.3.1) that

$$\frac{W_{\tau'_{L_t^p}} - W_{\tau'_{L_s^p}}}{\sqrt{t-s}} \equiv \frac{\sum_{i=0}^{\tau'_{L_t^p} - \tau'_{L_s^p} - 1}\left(W_{i+1+\tau'_{L_s^p}} - W_{i+\tau'_{L_s^p}}\right)}{\sqrt{t-s}} \underset{t-s\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(0,1).$$

Note that

$$-\frac{W^*}{\sqrt{t-s}} \overset{d}{\sim} \inf_{u\in\left[\tau_{L_t^p}, \tau_{L_t^p}+1\right]}\frac{W_u - W_{\tau_{L_t^p}}}{\sqrt{t-s}} \leq \frac{W_{\tau'_{L_t^p}} - W_{\tau_{L_t^p}}}{\sqrt{t-s}} \leq \sup_{u\in\left[\tau_{L_t^p}, \tau_{L_t^p}+1\right]}\frac{W_u - W_{\tau_{L_t^p}}}{\sqrt{t-s}} \overset{d}{\sim} \frac{W^*}{\sqrt{t-s}},$$

where the equivalence in distribution follows from the strong Markov property of brownian motion and $W^* := \sup_{u\in[0,1]} W_u$. Since $\mathbb{P}(W^* < \infty) = 1$ we have that $\frac{W_{\tau'_{L_t^p}} - W_{\tau_{L_t^p}}}{\sqrt{t-s}} \underset{t-s\to\infty}{\longrightarrow} 0$ a.s.. Similarly, $\frac{W_{\tau'_{L_s^p}} - W_{\tau_{L_s^p}}}{\sqrt{t-s}} \underset{t-s\to\infty}{\longrightarrow} 0$ a.s.. Hence, from Slutsky's theorem (see e.g. Hayashi (2000), Lemma 2.4), it follows that

$$\frac{W_{\tau_{L_t^p}} - W_{\tau_{L_s^p}}}{\sqrt{t-s}} \underset{t-s\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(0,1),$$

and a further application of Slutsky's theorem delivers

$$\frac{W_{\tau_{L_t^p}} - W_{\tau_{L_s^p}}}{\sqrt{L_t^p - L_s^p}} \equiv \frac{W_{\tau_{L_t^p}} - W_{\tau_{L_s^p}}}{\sqrt{t-s}} \frac{\sqrt{t-s}}{\sqrt{L_t^p - L_s^p}} \xrightarrow[t-s\to\infty]{d} \mathcal{N}\left(0, \mu_\tau\right).$$

Since, from the proof of Theorem 7 we know that $\log P_t$ has the same law as $H_{\tau_{L_t^p}}$ (see equation (42)) defined in equation (35), the conclusion of the proposition follows. ■

Equation (45) implies that, at low frequency, the log return process on the calendar time scale is characterised by stochastic volatility, and that the driver of time variation in volatility is the number of trades that occur between time t and s. Moreover, the fact that log returns are Gaussian, after a stochastic time change with respect to number of trades, is exactly the empirical finding of Ané and Geman (2000). Last but not least, this result implies that periods of high trading activity will tend to coincide with periods of increased return volatility and is consistent with the Wall St. wisdom that "it takes volume to move the price" (since, at low frequency, volume of trade is simply proportional to the number of trades).

The *ultra*-low frequency result in equation (46) arises due to the fact that, at this frequency, the number of trades per time interval converges, hence the stochastic volatility driven by the number of trades disappears (hence at this frequency fundamental and price volatility coincide as e.g. in Bernhardt and Taub (2008)). This finding is consistent with the fact that volatility clustering is, in the data, very evident at high and medium frequency, but typically harder to detect at extremely low frequency.

4 The Equilibrium Determinants of Liquidity and Volatility

Based on the results of the previous section, we can now analyse how the degree of asymmetric information and market frictions affect the equilibrium market liquidity and volatility, and how these quantities would be affected by the introduction of a Tobin Tax.

4.1 Equilibrium liquidity

Kyle (1985) defines a liquid market as one in which: *a*) the costs of trading small amounts are themselves small (bid-ask spreads are small) i.e. the market is *tight*; *b*) the costs of trading large amounts are small (big trades don't cause large price movements) i.e. the market is *deep*; *c*) discrepancies between prices and true values of assets are small and are corrected quickly i.e. the market is *resilient*.

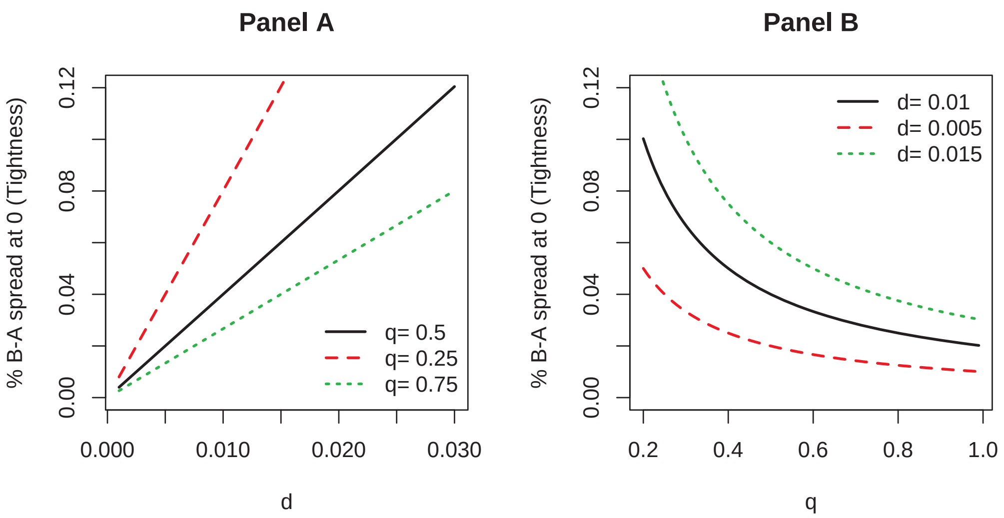

Figure 3: Market tightness. Bid-ask spread as a percentage of the market maker's estimate of the fundamental value as the order size approaches zero, i.e. $\frac{2q\delta}{q^2-\delta^2}$, as a function of δ (Panel A) and q (Panel B).

In our model, tightness, depth, and resilience are all determined in equilibrium, and they are all functions (that can be expressed in closed form) of adverse selection in the market (pinned down by the parameter $1-q$) and the magnitude of market frictions (embodied by the parameter δ). Moreover, an increase in the parameter δ can be interpreted as analogous to the introduction of a proportional financial transaction tax – aka Tobin Tax – of the type implemented in several countries and currently being under discussion within the European Union.

The tightness of the market can be obtained from the ask and bid price schedules in equations (13) and (14) of Proposition 4 as the order size approaches zero. The resulting percentage bid-ask spread (as a percentage of the market maker's estimate of the fundamental value) is equal to $\frac{2q\delta}{q^2-\delta^2}$ and is depicted in Figure 3. Note that, in our setting, the bid-ask spread is a function of only the degree of adverse selection $(1-q)$ and the order processing cost (δ). This is consistent with the empirical literature that finds that about 86-100% of the spread is generated by these two forces (see e.g. Stoll (1989), George, Kaul, and Nimalendran (1991), and Huang and Stoll (1996)), with the remaining fraction (if any) being driven by inventory costs. Panel A of Figure 3 shows that, as the degree of market friction δ increases, the bid-ask spread becomes wider, hence market tightness is decreasing in δ. This is simply due to the fact that increasing δ the trading cost incurred by the market maker becomes larger, hence, in order to compensate for this, the mark-up on the market maker valuation needed to break even in a trade increases. More interestingly, the rate at which market

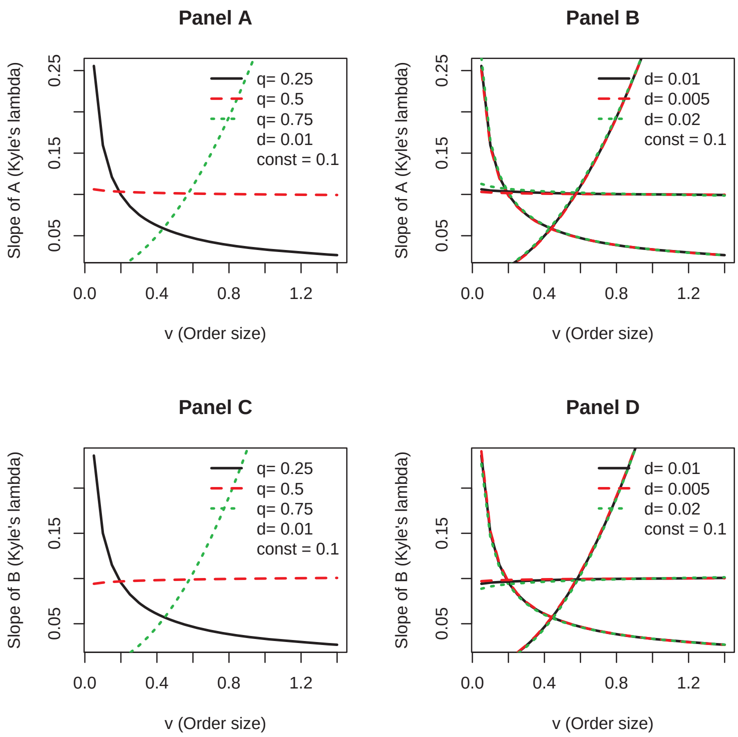

Figure 4: Inverse market depth i.e. Kyle's λ. Panels A and B depict the slope of the ask price schedule, normalised by the market maker valuation, i.e. $\frac{q}{1-q}\alpha(v^+)^{\frac{2q-\delta-1}{1-q}}$ as a function of the order size for different q, and different q and δ, respectively (Panel B considers the same values for q as in Panel A but adds perturbations to the value of δ). Panels C and D depict the analogous quantities for the bid price schedule i.e. $\frac{q}{1-q}\beta(v^-)^{\frac{2q+\delta-1}{1-q}}$. In all panels the constants α and β are fixed to the same value equal to 0.01.

tightness decreases in δ is higher when there are more informed traders (q is small) i.e. when the adverse selection problem faced by the market maker is more severe. Panel B shows that the tightness increases as the share of uninformed agents increases, since the degree of adverse selection in the market is reduced. These results imply that the introduction of a Tobin Tax would: *a*) reduce market tightness; *b*) exacerbate the adverse selection problem from the market maker (or limit order traders) perspective; and *c*) have more severe effects in markets with a high degree of adverse selection i.e. markets already characterised by low tightness.

The market depth can be elicited from the first derivative with respect to the order size of the ask and bid price schedules in equations (13) and (14) of Proposition 4, and is summarised in Figure 4. These derivatives (that in the figure are normalised by the market maker's valuation) are analogous to Kyle's lambda i.e. they represent the sensitivity of prices to order flows, and are thus inversely related to market depth. The first important thing to notice is that, in our setting, market depth is generally not constant – it is instead a function of the order size.[16] This is consistent with the empirical finding of Keim and Madhavan (1996) that the price impact per unit trade is itself a function of the order size (see also Loeb (1983) and Kavajecz (1999)).

Panels A and C show that there is a q^* threshold such that the market depth is increasing in the order size for $q < q^*$ and decreasing in the order size for $q > q^*$.[17] This is due to the fact that when q is high most of the traders are of the uninformed type hence, in this case, the price will be more likely to deviate substantially from the fundamental value. Therefore, the potential loss that the market maker would incur executing a large informed order is high. On the contrary, when q is low, most traders are of the informed type, and the price will be unlikely to deviate substantially from the fundamental value. Hence, the market maker's potential losses from executing a large order are substantially smaller. Given these considerations, the market maker chooses a decreasing or increasing market depth depending on the value of q.

Panels B and D show that, in the empirically relevant parameter range,[18] variations in δ have a very small effect on market depth. Hence, the introduction of a Tobin Tax is not likely to affect this dimension of liquidity. This result is intuitive given that the concept of market depth is about the relative willingness of executing small vs. large orders, and this willingness is unlikely to be substantially affected by a *proportional*, and small, trade tax. Moreover, this result, taken together with the observation that trading costs have a large impact on market tightness, suggests that the degree of asymmetric information in the market could be better inferred empirically form its depth rather than the tightness.

The degree of market resilience can be inferred combining the trade-by-trade market maker's valuation update function in equation (15), with the limiting number of trades per unit of time, $\lim_{t\to\infty} L_t^p/t \equiv 1/\mu_\tau$, in equation (44). The former has an half-life of deviations from the fundamental value – on the trade-by-trade time scale – equal to $\log .5/\log q$ i.e. it is decreasing in q and unaffected by δ.[19] Scaling this quantity by the number of trades per unit of time, we obtain the half-life of deviations from the fundamental value on the calendar time scale i.e.

$$\frac{\log 1/2}{\log q}\mu_\tau \tag{47}$$

where μ_τ is the (limiting) expected inter-arrival time of trades defined in equation (43).

[16] The derivative of the ask price with respect to the order size is constant if and only if $q = .5\,(1+\delta)$, and the one of the bid price is constant if and only if $q = .5\,(1-\delta)$.

[17] The q^* threshold is equal to $1/2 + .5\delta$ at ask and $1/2 - .5\delta$ at bid.

[18] See e.g. Stambaugh (2014) AFA presidential address.

[19] Rewriting equation (15) in deviation from the fundamental value, one obtains an AR(1) process for the deviation of the market maker's valuation from the fundamental value, with autoregressive coefficient q and shock proportional to the deviation of the i-th trader's valuation from the fundamental one. Hence, the h-period ahead impulse response function of a valuation shock is equal to q^h times the shock, delivering the half-life on the trade-by-trade time scale reported above.

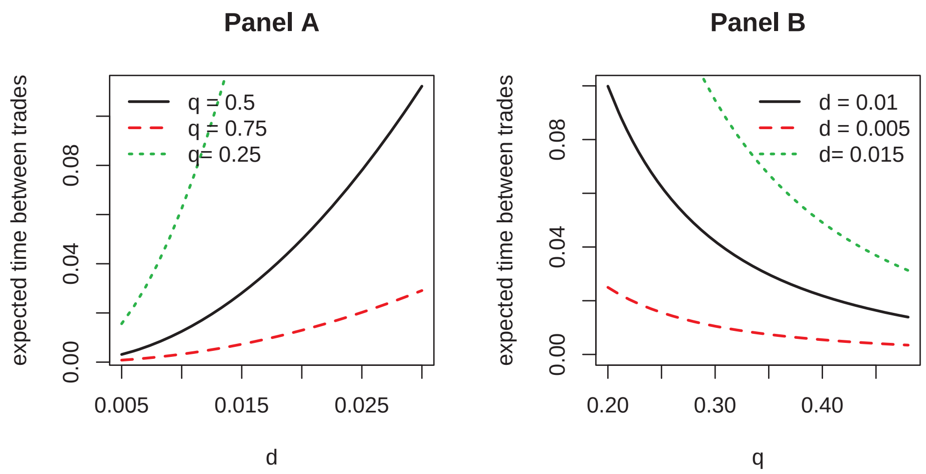

Figure 5: Expected inter-arrival time of trades (μ_τ, defined in equation (43)) as a function of δ (Panel A) and q (Panel B).

Since μ_τ is itself a function of q and δ, and resilience will inherit some of its properties (in particular with respect to δ), it is useful to understand first how the former varies when parameters change.

Figure 5 depicts the expected inter-arrival time of trades as a function of δ (Panel A) and q (Panel B). As one would intuit, the inter-arrival time is increasing in the transaction friction δ (Panel A). This is simply due to the fact that bid-ask spreads are widening in this quality, hence reducing the fraction of potential traders that, upon arrival, will decide to trade (since an increase in this quantity widens the no trade region of informed, as well as uninformed, traders). This is consistent with Umlauf (1993) that documents that the introduction of a Tobin Tax in the Swedish stock market in the 80's induced a reduction in turnover i.e. in the average number of trades per unit time, $1/\mu_\tau$.

More interestingly, the marginal effect of an increase in δ is larger when q is lower, that is when the market maker faces an higher degree of adverse selection. This is due to the fact that, as outlined before, the rate at which market tightness decreases in δ is higher when there are more informed traders (q is small). Panel B makes also clear that the expected inter-arrival time is decreasing in q. This is due to the fact that, as the degree of adverse selection is reduced, the market maker is more willing to trade, hence she increases market tightness (see Panel B of Figure 3), therefore increasing the share of potential traders that, upon arrival, choose to trade. Since the half-life of deviations of the specialist's valuation from the fundamental value in equation (47) depends upon δ only through μ_τ, the behaviour of resilience as a function of δ mimics the one of the expected inter-arrival time. Hence, resilience is decreasing in the degree of market friction δ, and this effect is more pronounced

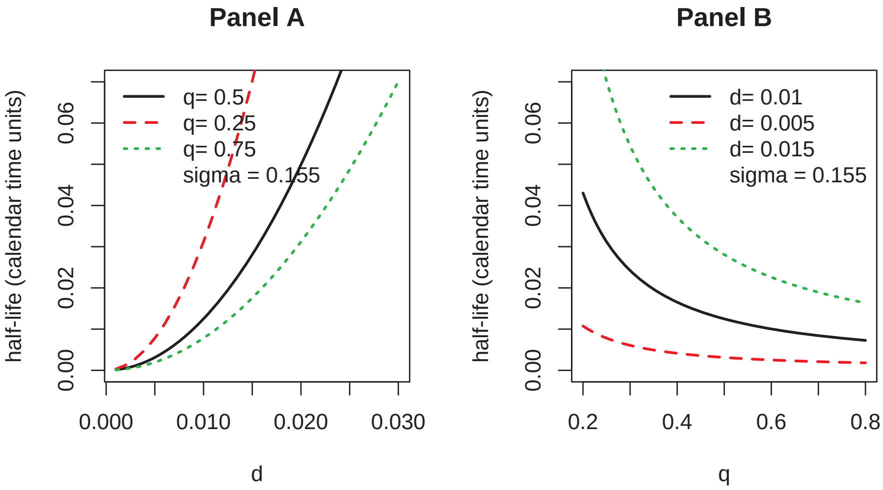

Figure 6: Market resilience. Half life of the market maker's valuation update in calendar time (i.e. inverse market resilience) defined in equation (47).

when the degree of adverse selection in the market is high (i.e. q is low) as depicted in Panel A of Figure 6.

The effect of a change in q on market resilience results from two counteracting forces. On one hand, the speed of the trade-by-trade valuation update of the market maker in equation (15) is accelerated as the share of informed agents increase (i.e. when q decreases). Hence, on the trade time scale, half-life reduces and resilience increases. On the other hand, in response to an increase in the degree of adverse selection, the specialist dealer reduces market tightness (see Panel B of Figure 3). This in turn increases the average time between trades μ_τ (see Panel B of Figure 5), hence it increases the calendar time half-life in equation (47), therefore reducing resilience. The net effect of these two opposing mechanisms, depicted in Panel B of Figure 6, is dominated by the adverse selection motive. That is, the calendar time half-life is decreasing, and resilience increasing, in q.

4.2 Equilibrium prices and volatility on different time scales

We have already shown in equation (19), reported below for the reader's convenience, that, at very high frequency, there is an equilibrium relationship between log returns and movements in volume:

$$\log \frac{P_{t+s}}{P_t} = \sum_{i=L_t}^{L_{t+s}} \left\{ \log \left(1 + \xi_i \left| V_{\tau_i} - V_{\tau_{i-1}} \right|^{\gamma_i} \right) + \log c_{1,i} + \log c_{2,i-1} \right\} \tag{48}$$

where c, γ and ξ are defined in equation (18). This equation implies that, at high frequency, the volatility of log returns is stochastic and is a function of $\left|V_{\tau_i} - V_{\tau_{i-1}}\right|$.

Note that the above equilibrium result is consistent with the seminal works of Epps and Epps (1976) and Tauchen and Pitts (1983) on the price-volume relationship and the empirical findings of (among others) Gallant, Rossi, and Tauchen (1992), Andersen (1996), and Chan and Fong (2000), that find a strong (often non-linear) link between volume of trade and price movements and between volume and price volatility.[20] Furthermore, equation (19) implies that if $|\xi_i| \left|V_{\tau_i} - V_{\tau_{i-1}}\right|^{\gamma_i}$ is small (i.e. if the typical transaction size is small), than a Taylor expansion would yield a power law relationship between order size and price growth rates. This is coherent with the empirical findings of Farmer and Lillo (2004) and Farmer, Lillo, and Mantegna (2003) that identify a log-linear relationship between gross price growth and changes in volume. On the other hand, if the typical transaction size is large (i.e. if $|\xi_i| \left|V_{\tau_i} - V_{\tau_{i-1}}\right|^{\gamma_i}$ is large), a log-log relationship between gross price growth and volume changes holds, which is consistent with the empirical findings of Potters and Bouchaud (2003).

As the intensity of arrivals approaches infinity – i.e. the medium time frequency – the equilibrium price process is characterised in Theorem 7 and Corollary 1. In particular, the limiting trade-by-trade variance of gross returns is $\frac{\delta^2(1-q^2)}{q^2-\delta^2}$. This quantity is depicted in Panels A and B of Figure 7 as a function, respectively, of δ and q. The figure shows that trade-by-trade volatility is increasing in the degree of trading friction δ, and that the marginal effect of an increase in δ is stronger when the degree of adverse selection faced by the market maker is high (i.e. when q is low). This behaviour is consistent Jones and Seguin (1997) that studies the effect of the lowering of the 1975 negotiated commission on the U.S. national stock exchange, and finds that the reduction in transaction cost is associated with a decline in stock market volatility. Moreover, Panel B shows that this variance is also decreasing in q. These behaviours are quite intuitive: as δ (q) increases (decreases) the market maker reduces market tightness (and resilience), and this mechanically increases volatility of trade prices.

Note that the trade-by-trade constant volatility does *not* imply constant volatility on the calendar time scale, since: *a*) trade times – hence number of trades in a given time period – are stochastic and endogenous, and *b*) prices (from equation (25) and Corollary 1) are serially correlated. Indeed, equation (45) in Proposition 10 shows that at low frequency the variance (between time s and t) is stochastic and given by $\left(L_t^p - L_s^p\right)\sigma^2\mu_\tau$ – that is, number of trades is the driver of stochastic volatility at this frequency as found in the empirical works of Jones, Kaul, and Lipson (1994).[21] Or, more precisely, in our setting low frequency

[20]See also Hausman, Lo, and MacKinlay (1992) and, for a survey of the empirical studies on the price, volume, and volatility relation, Karpoff (1987).

[21]See also Dufour and Engle (2000).

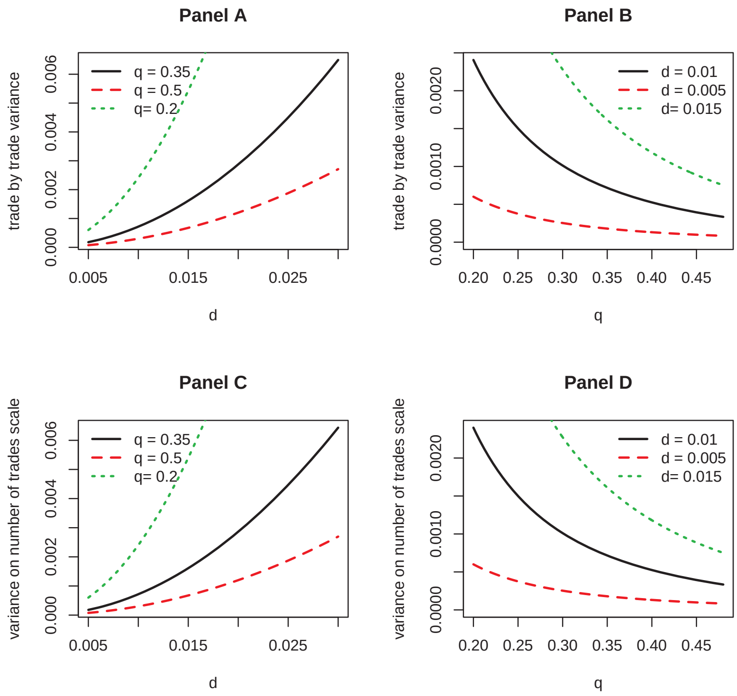

Figure 7: Limiting Variances. Variance on the trade by trade (upper panels) and number of trades (lower panels) time scales as a function of δ (left panels) and q (right panels).

log returns follow a Brownian motion time changed by the number of trades process. This is analogous to the empirical finding of Ané and Geman (2000) that find empirically that the distribution of log returns *conditional* on the number of trades is Gaussian and has constant volatility – this is precisely our result in equation (45) of Proposition 10, and the resulting (low frequency) constant volatility on the trade time scale ($\mu_\tau \sigma^2$) is depicted in Panels C and D of Figure 7. The two Panels show that the volatility of the price process, scaled by the (square root of) the number of trades has an almost identical behaviour as the trade-by-trade volatility in Panels A and B. Moreover, the rationale of this behaviour is analogous to the one described above for Panels A and B. One thing worth stressing is that the variances in the upper and lower panels of the figure, although very similar, are not identical – this discrepancy is due to the equilibrium autocorrelation of log returns.

Note that the above results on the equilibrium volatility have been obtained in a setting in which the fundamental was assumed to have constant volatility. This suggests that, if the fundamental volatility where to be time varying, the endogenous stochastic volatility mechanism outlined in our model would amplify this time variation.

4.3 Liquidity and volatility co-movements

We have seen above that the deep parameters of the model – namely, the degree of adverse selection on the market (measured by $1-q$) and the degree of other transaction frictions (captured by the parameter δ) – have first order effects on the equilibrium liquidity and volatility of the market that are consistent with the empirical literature. Therefore, a natural question is whether changes in these market fundamental characteristics can explain the co-movements of the liquidity measures and volatility documented in the empirical literature.

Indeed, the equilibrium liquidity measures discussed in Section 4.1 are not only individually, but also *jointly* consistent with the empirical evidence on the co-movement of different liquidity measures and trading activity. For instance, Dufour and Engle (2000) document a systematic comovement of the time duration between transactions, the price impact of trades, and the speed of price adjustment to trade-related information, and interpret the resulting reduction in liquidity as being induced by an increased presence of informed traders. In our setting, a change in the degree of adverse selection in the market (measured by $1-q$), manifests itself via exactly these joint changes in the behaviour of equilibrium liquidity.[22] Furthermore, in our model, an increase in adverse selection alsos cause an increase in volatility on both the trade-by-trade and number of trades time scale as illustrated in Figure 7. That is, our model is capable of generating joint liquidity dry-ups and volatility spikes in response to an increase in the degree of market adverse selection as, for instance, during the subprime crises.

Moreover, in our equilibrium characterisation, an increase in trading costs – as e.g. the introduction of a Tobin Tax or an increase in trading fees – reduces both market tightness and resilience, and increases volatility on the various time scales, as documented empirically e.g. by Jones and Seguin (1997), Jones, Kaul, and Lipson (1994) and Umlauf (1993). And this effect, as outlined in previous sections, is stronger in markets characterised by a higher degree of adverse selection.

5 Conclusion

This paper develops a tractable, asymmetric information based, equilibrium trading model, in which the distribution of the prices process, its volatility, the limit order book, the trading activity, as well as the various dimensions of market liquidity, are all characterised as functions of fundamental (trading and informational) frictions. The results derived provide micro-foundations for, and a rationalisation of, a large set of empirical findings including

[22]E.g. Dufour and Engle (2000) find that "as the time duration between transactions decreases, the price impact of [large] trades, the speed of price adjustment to trade-related information [...] all increase," and exactly this joint behaviour would arise in our setting in response to an increase in q.

the presence of (stochastic) volatility clustering, and a price volatility, volume, and trading activity link. Moreover, the framework developed constitute a natural laboratory for the analysis of the equilibrium impact of a Tobin tax, and delivers predictions consistent with the empirical evidence on this topic, as well as novel insights.

Methodologically, the multiple time scales and the limiting characterisation approach of the corresponding market equilibria, developed in this paper, could also be extended (with appropriate modifications) to study very different economic problems: e.g. from the effect of high frequency trading in financial markets, to the modelling of time and state contingent price setting in sticky prices, wages, and information, macroeconomic models.

Furthermore, our characterisation of the equilibrium price process, liquidity, trading activity, and volatility, at different frequencies, and as a function of the fundamental trading and informational frictions, naturally opens two important directions for future research.

First, our asymptotic characterisations, that are in nature of the market microstructure invariance type (see e.g. Kyle and Obizhaeva (2011a, 2001b)), raise a natural question: at what (calendar time) speed do the equilibrium processes converge (on the different time scales considered) to the equilibria that we have derived? These speeds of convergence should be functions of the fundamental trading (δ) and informational (q) frictions on the market. Hence these frictions should, also through this channel, influence the (calendar) time series of market dynamics and risk. The task of analysing these speeds of convergence is complicated by the fact that an obvious metric for quantifying the speed of convergence between distributions is not readily available. Nevertheless, a potentially promising metric is the relative entropy, as a function of the fundamental frictions, between the equilibrium price distributions at any given frequency and the next, lower frequency, distribution. For instance, the half-life of the relative entropy (or generalised variance) discrepancy would be a relevant statistic to construct in order to understand how financial risk is generated.[23]

Second, our closed form characterisations of the price process, liquidity, trading activity, limit order book, and volatility, as a function of the fundamental frictions, offer a natural approach for the investigation of the empirical relevance of these channels in driving financial market dynamics, and for the estimation of the fundamental market characteristics that generates them. Moreover, for a richer empirical analysis, the framework derived in this paper could be generalized to accommodate time varying fundamental volatility, time varying degree of adverse selection, and time varying trading costs (as e.g. the time varying margins – aka "haircuts" – studied in Brunnermeier and Pedersen (2009)).

Both of the above extensions are promising, although demanding, tasks, and we defer them to future work.

[23]For an application of relative entropy divergences to the analysis of risk measures see e.g. Julliard and Ghosh (2012) and Ghosh, Julliard, and Taylor (2011).

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A Appendix

A.1 Additional Proofs and Lemmas

Proof of Proposition 4. Given the market maker's indifference conditions (9) and (10) and Lemma 1, it follows from Bayes rule that

$$\begin{aligned}A_t\left(v^+\right)(1-\delta) &= \sum_{i=1}^{\infty}\mathbf{1}_{\{i=1+L_{t-}\}}\left\{\mathbb{P}\left[\tilde{I}_i|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i}\right]\mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i},\tilde{I}_i\right]+\right.\\ &\left.\mathbb{P}\left[\tilde{U}_i|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i}\right]\mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i},\tilde{U}_i\right]\right\}\Big|_{\tilde{v}_i=v^+,\tau_i=t},\\ &=\left(1-q_t\left(v^+\right)\right)\left[A_t\left(v^+\right)+vA_t'\left(v^+\right)\right]+q_t\left(v^+\right)X_t\left(v^+\right) \end{aligned} \tag{49}$$

$$\begin{aligned}B_t\left(v^-\right)(1+\delta) &= \sum_{i=1}^{\infty}\mathbf{1}_{\{i=1+L_{t-}\}}\left\{\mathbb{P}\left[\tilde{I}_i|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i}\right]\mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i},\tilde{I}_i\right]+\right.\\ &\left.\mathbb{P}\left[\tilde{U}_i|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i}\right]\mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i},\tilde{U}_i\right]\right\}\Big|_{\tilde{v}_i=v^-,\tau_i=t}\\ &=\left(1-q_t\left(-v^-\right)\right)\left[B_t\left(v^-\right)+vB_t'\left(v^-\right)\right]+q_t\left(-v^-\right)X_t\left(-v^-\right) \end{aligned} \tag{50}$$

where $\tilde{I}_i$ ($\tilde{U}_i$) denotes the event of the i-th trader being informed (uninformed) and

$$\begin{aligned}q_t(v) &= \sum_{i=1}^{\infty}\mathbf{1}_{\{i=1+L_{t-}\}}\,\mathbb{P}\left[\tilde{U}_i|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i}\right]\Big|_{\tilde{v}_i=v,\tau_i=t}\\ X_t(v) &= \sum_{i=1}^{\infty}\mathbf{1}_{\{i=1+L_{t-}\}}\,\mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i},\tilde{U}_i\right]\Big|_{\tilde{v}_i=v,\tau_i=t}\end{aligned}$$

where $q_t(v)$ is the probability of the time t trader being uninformed and $X_t(v)$ is the time t market maker valuation given that the current trader is uninformed. Note that in the above equation the market maker uses the trader's valuation of the asset (from Lemma 1) only in the case of the trader being informed.

Recall that, from Remark (1) and (2), we have that $\tilde{\mathcal{H}}_i^M = \sigma\left\{\{\tilde{v}\}_{j=0}^i, \{\tau_j\}_{j=0}^i\right\}$ and $\mathcal{H}_i = \sigma\left\{\{v\}_{j=0}^i, \{\theta_j\}_{j=0}^i\right\}$. This observation allows us to rewrite the marker maker's probability of the trader being uninformed as $\mathbb{P}\left[\tilde{U}_i|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i}\right]=\mathbb{P}\left[\tilde{U}_i|\tilde{\mathcal{H}}_{i-1}^M,\tilde{v}_i,\tau_i, N_{\tau_i}=L_{\tau_i}\right]$.

Note that from Bayes rule, for any $C\in\mathcal{B}(\mathbb{R})$ we have

$$\mathbb{P}\left[\tilde{U}_i|\tilde{\mathcal{H}}_{i-1}^M,\tilde{v}_i\in C,\tau_i, N_{\tau_i}=L_{\tau_i}\right]=\frac{\mathbb{P}\left[\tilde{U}_i|\tilde{\mathcal{H}}_{i-1}^M,\tau_i, N_{\tau_i}=L_{\tau_i}\right]}{\frac{\mathbb{P}\left[\tilde{v}_i\in C|\tilde{\mathcal{H}}_{i-1}^M,\tau_i,N_{\tau_i}=L_{\tau_i}\right]}{\mathbb{P}\left[\tilde{v}_i\in C|\tilde{\mathcal{H}}_{i-1}^M,\tau_i,N_{\tau_i}=L_{\tau_i},\tilde{U}_i\right]}}. \tag{51}$$

From A4 and the fact that

$$\left\{\tilde{\mathcal{H}}_{i-1}^M\vee\sigma\left\{\tau_i,\tilde{U}_i\right\}\right\}\cap\left\{N_{\tau_i}=L_{\tau_i}\right\}=\left\{\mathcal{H}_{i-1}\vee\sigma\left\{\theta_i,U_i\right\}\right\}\cap\left\{N_{\tau_i}=L_{\tau_i}\right\} \tag{52}$$

it follows that Equation (51) simplifies to

$$\mathbb{P}\left[\tilde{U}_i|\tilde{\mathcal{H}}_{i-1}^M, \tilde{v}_i \in C, \tau_i, N_{\tau_i} = L_{\tau_i}\right] = \mathbb{P}\left[\tilde{U}_i|\tilde{\mathcal{H}}_{i-1}^M, \tau_i, N_{\tau_i} = L_{\tau_i}\right]$$

since $\mathbb{P}\left[\tilde{v}_i \in C|\tilde{\mathcal{H}}_{i-1}^M, \tau_i, N_{\tau_i} = L_{\tau_i}\right] = \mathbb{P}\left[\tilde{v}_i \in C|\tilde{\mathcal{H}}_{i-1}^M, \tau_i, N_{\tau_i} = L_{\tau_i}, \tilde{U}_i\right]$.

Finally, from Assumption A3 and the equality (52), we have that the arrival of an uninformed agent, $\tilde{U}_i$, is independent from $\tilde{\mathcal{H}}_{i-1}$ and τ_i, therefore

$$q_t(v) := \mathbb{P}\left[\tilde{U}_i|\tilde{\mathcal{H}}_{i-1}, \tau_i, N_{\tau_i} = L_{\tau_i}\right] = q.$$

Using equality(52), and the fact that the signal received by the uniformed trader is conditionally independent (A2), we have that $X_t(v) = Z_t^M$ since

$$\begin{aligned}
X_t(v) &= \sum_{i=1}^{\infty} \mathbf{1}_{\{i=1+L_{t-}\}}\, \mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_i^M, N_{\tau_i} = L_{\tau_i}, \tilde{U}_i\right]\Big|_{\tilde{v}_i=v,\tau_i=t} \\
&= \sum_{i=1}^{\infty} \mathbf{1}_{\{i=1+L_{t-}\}}\, \mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_{i-1}^M, \tilde{v}_i, \tau_i, N_{\tau_i} = L_{\tau_i}, \tilde{U}_i\right]\Big|_{\tilde{v}_i=v,\tau_i=t} \\
&= \sum_{i=1}^{\infty} \mathbf{1}_{\{i=1+L_{t-}\}}\, \mathbb{E}\left[\mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_{i-1}^M, S_{\tau_i}, \tau_i, N_{\tau_i} = L_{\tau_i}, \tilde{U}_i\right]|\tilde{\mathcal{H}}_{i-1}^M, \tilde{v}_i, \tau_i, N_{\tau_i} = L_{\tau_i}, \tilde{U}_i\right]\Big|_{\tilde{v}_i=v,\tau_i=t} \\
&= \sum_{i=1}^{\infty} \mathbf{1}_{\{i=1+L_{t-}\}} \mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_{i-1}^M, N_t = L_t\right] = \sum_{i=0}^{\infty} \mathbf{1}_{\{i=L_{t-}+1\}} \mathbb{E}\left[e^{D_T}\middle|\, \mathcal{G}_{\tau_{i-1}}^M, N_t = L_t\right] \\
&= \sum_{i=0}^{\infty} \mathbf{1}_{\{i=L_{t-}+1\}} Z_{\tau_{i-1}}^M
\end{aligned}$$

Therefore Equations (49) and (50) simplify to the following ordinary differential equations

$$\begin{aligned}
A_t(v)(1-\delta) &= (1-q)\left[A_t(v) + vA_t'(v)\right] + q\sum_{i=0}^{\infty} \mathbf{1}_{\{i=L_{t-}+1\}} Z_{\tau_{i-1}}^M, \\
B_t(v)(1+\delta) &= (1-q)\left[B_t(v) + vB_t'(v)\right] + q\sum_{i=0}^{\infty} \mathbf{1}_{\{i=L_{t-}+1\}} Z_{\tau_{i-1}}^{\underline{M}}.
\end{aligned}$$

These, up to a generic constant, have three solutions each, but only one solution per equation satisfies conditions C2 and C3. These solutions are the ones in equations (13) and (14). These solutions clearly satisfy conditions C2-C5, and C1 is satisfied because $\sum_{i=0}^{\infty} \mathbf{1}_{\{i=L_{t-}+1\}} Z_{\tau_{i-1}}^M$ is a *cádlág* process. ■

Proof of Lemma 3. The proof is by induction on i.

I. $i = 1$. Then

$$\mathbb{P}\left[d_1^{tr} \le x|\mathcal{H}_0, \theta_1\right] = q\mathbb{P}\left[d_1^{tr} \le x|\mathcal{H}_0, \theta_1, U_1\right] + (1-q)\,\mathbb{P}\left[d_1^{tr} \le x|\mathcal{H}_0, \theta_1, I_1\right]$$

since, due to A3, $\mathbb{P}[U_1|\mathcal{H}_0, \theta_1] = q$. From equation (24) and Remark 3 it follows that

$\mathbb{P}\left[d_1^{tr} \le x|\mathcal{H}_0, \theta_1, U_1\right] = \mathbb{P}\left[d_1^{tr} \le x|\mathcal{H}_0, \theta_1, I_1\right]$. Therefore

$$\mathbb{P}\left[d_1^{tr} \le x|\mathcal{H}_0, \theta_1\right] = \mathbb{P}\left[d_1^{tr} \le x|\mathcal{H}_0, \theta_1, I_1\right]. \tag{53}$$

On the other hand, we have

$$\begin{aligned}\mathbb{P}\left[d_1^{tr} \le x|\mathcal{H}_0, \theta_1, I_1\right] &= \mathbb{P}\left[D_0 + (D_{\theta_1} - D_0) \le x|\mathcal{H}_0, \theta_1, I_1\right] \\ &= \mathbb{P}\left[D_0 + (D_{\theta_1} - D_0) \le x|\mathcal{H}_0, \theta_1\right] \\ &= \mathbb{P}\left[d_0^{tr} + \varepsilon_{1,0} \le x|d_0^{tr}, \Delta_{1,0}\right]\end{aligned} \tag{54}$$

since $D_{\theta_1} - D_0$ is a Brownian motion increment and, due to assumption A3, this Brownian motion is independent of θ and I. Note also that Equations (53) and (54) imply

$$\mathbb{P}\left[d_1^{tr} \le x|\mathcal{H}_0, \theta_1\right] = \mathbb{P}\left[D_{\theta_1} \le x|\mathcal{H}_0, \theta_1\right]$$

II. Suppose the statement is true for $i = n$. Consider $i = n + 1$, then

$$\mathbb{P}\left[d_{n+1}^{tr} \le x|\mathcal{H}_n, \theta_{n+1}\right] = q\mathbb{P}\left[d_{n+1}^{tr} \le x|\mathcal{H}_n, \theta_{n+1}, U_{n+1}\right] + (1-q)\,\mathbb{P}\left[d_{n+1}^{tr} \le x|\mathcal{H}_n, \theta_{n+1}, I_{n+1}\right] \tag{55}$$

$$= \mathbb{P}\left[d_{n+1}^{tr} \le x|\mathcal{H}_n, \theta_{n+1}, I_{n+1}\right]. \tag{56}$$

since, due to A3, $\mathbb{P}\left[U_{n+1}|\mathcal{H}_n, \theta_{n+1}\right] = q$, and from Remark 3 we know that

$$\mathbb{P}\left[d_{n+1}^{tr} \le x|\mathcal{H}_n, \theta_{n+1}, I_{n+1}\right] = \mathbb{P}\left[d_{n+1}^{tr} \le x|\mathcal{H}_n, \theta_{n+1}, U_{n+1}\right].$$

we also have

$$\begin{aligned}\mathbb{P}\left[d_{n+1}^{tr} \le x|\mathcal{H}_n, \theta_{n+1}, I_{n+1}\right] &= \mathbb{P}\left[D_{\theta_{n+1}} \le x|\mathcal{H}_n, \theta_{n+1}, I_{n+1}\right] \\ &= \mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \le x|\mathcal{H}_n, \theta_{n+1}, I_{n+1}\right].\end{aligned}$$

Note that from assumption A3 $I_{n+1} \perp \mathcal{F}_T^{N,D,S} \vee (U_j)_{j \ne n+1}$ $\forall n$. Since $\mathcal{H}_n \vee \sigma\left\{\theta_{n+1}\right\} \vee \sigma\left\{D_{\theta_{n+1}} - D_{\theta_n}\right\} \vee \sigma\left\{D_{\theta_n}\right\} \subset \mathcal{F}_T^{N,D,S} \vee (U_j)_{j \ne n+1}$ we have

$$I_{n+1} \perp \mathcal{H}_n \vee \sigma\left\{\theta_{n+1}\right\} \vee \sigma\left\{D_{\theta_{n+1}} - D_{\theta_n}\right\} \vee \sigma\left\{D_{\theta_n}\right\}\ \forall n.$$

Therefore, from Proposition 6.8 of Kallenberg (2002), we have that, $\forall n$, $I_{n+1} \underset{\mathcal{H}_n \vee \sigma\{\theta_{n+1}\}}{\perp} \sigma\left\{D_{\theta_{n+1}} - D_{\theta_n}\right\} \vee \sigma\left\{D_{\theta_n}\right\}$, and from Proposition 6.6. of Kallenberg (2002) it follows that

$$\mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \le x|\mathcal{H}_n, \theta_{n+1}, I_{n+1}\right] = \mathbb{P}\left[D_{\theta_{n+1}} \le x|\mathcal{H}_n, \theta_{n+1}\right]. \tag{57}$$

Note that the above and Equation (56) imply $\mathbb{P}\left[d_{n+1}^{tr} \le x|\mathcal{H}_n, \theta_{n+1}\right] = \mathbb{P}\left[D_{\theta_{n+1}} \le x|\mathcal{H}_n, \theta_{n+1}\right]$.

Moreover

$$\mathbb{P}\left[d_{n+1}^{tr} \leq x|\mathcal{H}_n, \theta_{n+1}, I_{n+1}\right] = \mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_n, \theta_{n+1}\right]$$

$$= (1-q)\,\mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_n, \theta_{n+1}, I_n\right] + q\mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_n, \theta_{n+1}, U_n\right] \tag{58}$$

where the last equality simply follows from Bayes rule and that Assumption A3, and the fact that $\mathcal{H}_n \vee \sigma\{\theta_{n+1}\} \subset \mathcal{F}_T^{N,D,S} \vee (U_j)_{j\neq n+1}$, imply that $\mathbb{P}[U_n|\mathcal{H}_n, \theta_{n+1}] = q$.
Note that the I_n agent knows D_{θ_n}, therefore

$$\begin{aligned}
&\mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_n, \theta_{n+1}, I_n\right] = \mathbb{P}\left[d_n^{tr} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_n, \theta_{n+1}, I_n\right] \\
&= \mathbb{P}\left[d_n^{tr} + \sigma\left(W_{\theta_{n+1}} - W_{\theta_n}\right) + \mu\Delta_{n+1,n} \leq x|\mathcal{H}_n, \theta_{n+1}, I_n\right] \\
&= \mathbb{P}\left[d_n^{tr} + \sigma\left(W_{\theta_{n+1}} - W_{\theta_n}\right) + \mu\Delta_{n+1,n} \leq x|\mathcal{H}_n, \theta_{n+1}\right].
\end{aligned}$$

where the last equality follows from assumption A3 and the fact that $\mathcal{H}_n \vee \sigma\{\theta_{n+1}\} \vee \sigma\left\{W_{\theta_{n+1}} - W_{\theta_n}\right\} \subset \mathcal{F}_T^{N,D,S} \vee (U_j)_{j\neq n+1}$, hence we can use use once more Proposition 6.6 and 6.8 of Kallenberg (2002).
Let define $\tilde{W}_t^n := W_{\theta_n+t} - W_{\theta_n}$ and $\tilde{N}_t^n := N_{\theta_n+t} - N_{\theta_n}$. From Assumption A2, and the fact that W is a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$, we have $\mathcal{F}^{\tilde{W}^n} \underset{\mathcal{H}_n}{\perp} \mathcal{F}^{\tilde{N}^n}$, and $\mathcal{F}^{\tilde{W}^n} \perp \mathcal{H}_n$. Therefore, from Proposition 6.8 of Kallenberg (2002) the above is equivalent to $\mathcal{F}^{\tilde{W}^n} \perp \mathcal{F}^{\tilde{N}^n} \vee \mathcal{H}_n$. Since from Assumption A1 $\mathcal{F}^{\tilde{N}^n} \perp \mathcal{H}_n$, it follows from the definition of independence that $\mathcal{F}^{\tilde{N}^n} \vee \mathcal{F}^{\tilde{W}^n} \perp \mathcal{H}_n$. Since $\sigma\left\{W_{\theta_{n+1}} - W_{\theta_n}\right\} \vee \sigma\{\Delta_{n+1,n}\} \subset \mathcal{F}^{\tilde{N}^n} \vee \mathcal{F}^{\tilde{W}^n}$, the above independence and Proposition 6.8 of Kallenberg (2002) implies $\sigma\left\{W_{\theta_{n+1}} - W_{\theta_n}\right\} \underset{\sigma\{\Delta_{n+1,n}\}}{\perp} \mathcal{H}_n$. Thus, we have by Proposition 6.6 of Kallenberg (2002) that

$$\mathbb{P}\left[d_n^{tr} + \sigma\left(W_{\theta_{n+1}} - W_{\theta_n}\right) + \mu\Delta_{n+1,n} \leq x|\mathcal{H}_n, \theta_{n+1}\right] = \mathbb{P}\left[d_n^{tr} + \varepsilon_{n+1,n} \leq x|d_n^{tr}, \Delta_{n+1,n}\right]$$

where $\varepsilon_{n+1,n} := \mu\Delta_{n+1,n} + \sigma\sqrt{\Delta_{n+1,n}}\eta_{n+1,n}$ and $\eta_{n+1,n} \sim N(0,1)$ is independent of d_n^{tr} and $\Delta_{n+1,n}$ for all n. Therefore

$$\mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_n, \theta_{n+1}, I_n\right] = \mathbb{P}\left[d_n^{tr} + \varepsilon_{n+1,n} \leq x|d_n^{tr}, \Delta_{n+1,n}\right]. \tag{59}$$

To complete the characterisation of Equation (55) we now simplify $\mathbb{P}\left[d_{n+1}^{tr} \leq x|\mathcal{H}_n, \theta_{n+1}, U_{n+1}\right]$. Observe that

$$\begin{aligned}
&\mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_n, \theta_{n+1}, U_n\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_{n-1}, \theta_{n+1}, \theta_n, U_n, S_n\right]|\mathcal{H}_n, \theta_{n+1}, U_n\right] \\
&= \mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_{n-1}, \theta_{n+1}, \theta_n\right]
\end{aligned}$$

where the last equality is due to Assumptions A2 and A3. Using Propositions 6.6 and 6.8 of Kallenberg (2002) in the same fashion as above, we have

$$\mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_n, \theta_{n+1}, U_n\right] = \mathbb{P}\left[D_{\theta_n} + \varepsilon_{n+1,n} \leq x|\mathcal{H}_{n-1}, \theta_{n+1}, \theta_n\right] \tag{60}$$

where $\varepsilon_{n+1,n} := \mu\Delta_{n+1,n} + \sigma\sqrt{\Delta_{n+1,n}}\eta_{n+1,n}$ and $\eta_{n+1,n} \sim N(0,1)$ is independent of $\mathcal{H}_{n-1}, \theta_{n+1}, \theta_n$ and $\mathcal{F}^D_{\theta_n}$.

To complete the induction recall that in Assumption A2 $\mathcal{F}^W_T \underset{\mathcal{H}_{i-1}}{\perp} \mathcal{F}^N_T$. Since $\sigma\{\theta_n\} \vee \sigma\{\theta_{n+1}\} \subset \mathcal{F}^N_T$ we have $\mathcal{F}^W_T \underset{\mathcal{H}_{i-1}}{\perp} \sigma\{\theta_n\} \vee \sigma\{\theta_{n+1}\}$. Thus, from Proposition 6.8 and Corollary 6.7 of Kallenberg (2002) we have that $\mathcal{F}^W_T \vee \sigma\{\theta_n\} \underset{\mathcal{H}_{i-1},\sigma\{\theta_n\}}{\perp} \sigma\{\theta_{n+1}\}$, and since $\sigma\{D_{\theta_n}\} \subset \mathcal{F}^W_T \vee \sigma\{\theta_n\}$, we have $\sigma\{D_{\theta_n}\} \underset{\mathcal{H}_{i-1},\sigma\{\theta_n\}}{\perp} \sigma\{\theta_{n+1}\}$. Therefore, for any χ

$$\begin{aligned}
\mathbb{P}\left[D_{\theta_n} \leq \chi|\mathcal{H}_{n-1}, \theta_{n+1}, \theta_n\right] = \mathbb{P}\left[D_{\theta_n} \leq \chi|\mathcal{H}_{n-1}, \theta_n\right] &= \mathbb{P}\left[d^{tr}_n \leq \chi|\mathcal{H}_{n-1}, \theta_n\right] \\
&= (1-q)\sum_{j=1}^{n-1} q^{n-1-j}\mathbb{P}\left[d^{tr}_j + \varepsilon_{n,j} \leq x|d^{tr}_j, \Delta_{n,j}\right] + \\
&+ q^{n-1}\mathbb{P}\left[d^{tr}_0 + \varepsilon_{n,0} \leq x|d^{tr}_0, \Delta_{n,0}\right]
\end{aligned}$$

where the last two equalities are due to the induction assumption that also imply $\varepsilon_{n,j} := \mu\Delta_{n,j} + \sigma\sqrt{\Delta_{n,j}}\eta_{n,j}$, and $\eta_{n,j} \sim N(0,1)$ is independent of d^{tr}_j and $\Delta_{n,j}$ for all $j < n$.

Thus Equation (60) becomes

$$\begin{aligned}
&\mathbb{P}\left[D_{\theta_n} + D_{\theta_{n+1}} - D_{\theta_n} \leq x|\mathcal{H}_n, \theta_{n+1}, U_n\right] \\
&= (1-q)\sum_{j=1}^{n-1} q^{n-1-j}\mathbb{P}\left[d^{tr}_j + \varepsilon_{n,j} + \varepsilon_{n+1,n} \leq x|d^{tr}_j, \Delta_{n,j}, \Delta_{n+1,n}\right] + \\
&+ q^{n-1}\mathbb{P}\left[d^{tr}_0 + \varepsilon_{n,0} + \varepsilon_{n+1,n} \leq x|d^{tr}_0, \Delta_{n,0}, \Delta_{n+1,n}\right] \\
&= (1-q)\sum_{j=1}^{n-1} q^{n-1-j}\mathbb{P}\left[d^{tr}_j + \varepsilon_{n+1,j} \leq x|d^{tr}_j, \Delta_{n+1,j}\right] + q^{n-1}\mathbb{P}\left[d^{tr}_0 + \varepsilon_{n+1,0} \leq x|d^{tr}_0, \Delta_{n+1,0}\right]
\end{aligned}$$

since all the ε's are independent Gaussians.

Combining the above equation with equations (59), (58), and (56) yields

$$\begin{aligned}
\mathbb{P}\left[d^{tr}_{n+1} \leq x|\mathcal{H}_n, \theta_{n+1}\right] = (1-q)&\sum_{j=1}^{n} q^{n-j}\mathbb{P}\left[d^{tr}_j + \varepsilon_{n+1,j} \leq x|d^{tr}_j, \Delta_{n+1,j}\right] + \\
&+ q^n\mathbb{P}\left[d^{tr}_0 + \varepsilon_{n+1,0} \leq x|d^{tr}_0, \Delta_{n+1,0}\right].
\end{aligned}$$

By the principle of mathematical induction the proof is complete.

■

Proof of Corollary 1. To prove the corollary we need to compute the probability that, given that a trade occurred, it is at ask or bid. Let $\left(\mathcal{F}_t^W\right)$ be the natural filtration of W augmented in the usual way. Since W is a Brownian motion with respect to a (potentially) larger filtration, it is also a Brownian motion with respect to $\left(\mathcal{F}_t^W\right)$. Then,

$$\mathbb{P}\left[\sigma\left(W_{\tau_i}-W_{\tau_{i-1}}\right)-\frac{\sigma^2}{2}\left(\tau_i-\tau_{i-1}\right)=a\left(\frac{q}{\phi_{i-1}}+1-q\right)|\mathcal{F}_{\tau_{i-1}}^W\right]=\mathbb{P}\left[\sigma W_\tau-\frac{\sigma^2}{2}\tau=a\left(x\right)\right]_{x=\frac{q}{\phi_{i-1}}+1-q}$$

where the equality follows from the strong Markov property of Brownian motion, and where

$$\tau:=\inf\left\{t\geq 0:\,\sigma W_t-\frac{\sigma^2}{2}t\notin\left[b\left(x\right),\,a\left(x\right)\right]\right\}.$$

Note that $M_t:=\exp\left\{\sigma W_t-\frac{\sigma^2}{2}t\right\}$ is a martingale and $\tau\wedge s$ is a bounded stopping time for every fixed s. Thus, $\mathbb{E}M_{\tau\wedge s}=1$ by Doob's optional sampling theorem (see Revuz and Yor (1999) Th. 3.2 Ch. II). Since $M_{\tau\wedge s}$ is bounded for all s, we can use the dominated convergence theorem to obtain that $\mathbb{E}M_\tau=1$. Thus we have

$$\mathbb{P}\left[\sigma W_\tau-\frac{\sigma^2}{2}\tau=a\left(x\right)\right]=\frac{1-e^{b(x)}}{e^{a(x)}-e^{b(x)}}=\frac{q^2-\delta^2-q\left(q-\delta\right)x}{2q\delta x}$$

since M_τ can take only value $\exp\left\{a\left(x\right)\right\}$ or $\exp\left\{b\left(x\right)\right\}$. Hence ϕ_i has the stated distribution and the conditional moments follow from simple direct calculations. ■

Proof of Lemma 5. To show the continuity of the map over the set $\mathcal{C}$, we need to show that for any $f^n\to f\in\mathcal{C}$ in Skorokhod topology, we'll have $\mathcal{P}_2f^n\to\mathcal{P}_2f$ in Skorokhod topology. Due to Theorem VI.1.14 of Jacod and Shiryaev (2003), to establish the result it is enough to demonstrate that there exist a sequence of continuos functions $\rho^n:\mathbb{R}_+\to\mathbb{R}_+$ that are strictly increasing with $\rho^n\left(0\right)=0$ and $\lim_{t\to\infty}\rho^n\left(t\right)=\infty$, such that: $\sup_{t\in\mathbb{R}_+}\left|\rho^n\left(t\right)-t\right|\to 0$, and $\sup_{t\in\mathbb{R}_+}\left|\mathcal{P}_2f^n\left(\rho^n\left(t\right)\right)-\mathcal{P}_2f\left(t\right)\right|\to 0$.

Suppose that, for any $\tilde{\varepsilon}>0$ we can show that there exists $\bar{N}$ such that, for any $n\geq\bar{N}$, we have

$$L_T^n:=L_T^{f^n}=L_T^f, \tag{61}$$

$$\max_{i=0,\ldots,L_T^f}\left|\tau_i^n-\tau_i^f\right|\leq\frac{\tilde{\varepsilon}K_\tau}{4T} \tag{62}$$

$$\max_{i=0,\ldots,L_T^f}\left|g^n\left(\tau_i^n\right)-e^{f\left(\tau_i^f\right)}\right|<\tilde{\varepsilon} \tag{63}$$

where $\tau^n:=\tau^{f^n}$, $g^n:=g^{f^n}$. Under the above conditions, we can define

$$\rho^n\left(t\right):=\begin{cases}\frac{\tau_i^n-\tau_{i-1}^n}{\tau_i^f-\tau_{i-1}^f}\left(t-\tau_{i-1}^f\right)+\tau_{i-1}^n, & \text{for } t\in\left[\tau_{i-1}^f,\tau_i^f\right],\quad i=1,...,L_T^f+1,\\ t, & \text{for } t\geq T\end{cases}$$

with the convention that $\tau^n_{L^f_T+1}=\tau^f_{L^f_T+1}=T$. Note that this ρ^n is continuous and strictly increasing on $[0,T]$, $\rho^n(0)=0$, $\lim_{t\to\infty}\rho^n(t)=\infty$, and that

$$\sup_{t\in\mathbb{R}_+}|\rho^n(t)-t|=\max_{i=1,\dots,L^f_T+1}\left|\frac{\tau^n_i-\tau^n_{i-1}}{\tau^f_i-\tau^f_{i-1}}\left(t-\tau^f_{i-1}\right)+\tau^n_{i-1}-t\right|\le 4T\frac{\varepsilon}{K_\tau}<\tilde{\varepsilon}$$

and

$$\sup_{t\in\mathbb{R}_+}|\mathcal{P}_2f^n(\rho^n(t))-\mathcal{P}_2f(t)|=\max_{i=0,\dots,L^f_T}\left|g^n(\tau^n_i)-e^{f\left(\tau^f_i\right)}\right|<\tilde{\varepsilon},$$

since that both $\mathcal{P}_2f^n$ and $\mathcal{P}_2f$ are constant, respectively, on $\left(\tau^n_{i-1},\tau^n_i\right)$ and $\left(\tau^f_{i-1},\tau^f_i\right)$, as well as after $\tau^n_{L^f_T}$ and $\tau^f_{L^f_T}$. Thus, $\mathcal{P}_2$ would satisfy the convergence requirement if conditions (61)-(63) are fulfilled.

To show that conditions (61)-(63) are indeed satisfied for big enough n, consider

$$K_1=\frac{1}{2}\min\left\{\log\frac{q+\delta}{q-\delta};-\log\frac{q(1-\delta)}{q+\delta};\log\frac{q(1+\delta)}{q-\delta}\right\}.$$

Since the function f is continuos, for any i there exist strictly positive constants ε^l_i and ε^r_i such that:

- if $i\ge1$ and $f\left(\tau^f_i\right)=f\left(\tau^f_{i-1}\right)+a(c_{2,i-1})$ (i.e. the bound is crossed at the ask)

$$\min_{t\in\left[\tau^f_i-\varepsilon^l_i,\ \tau^f_i+\varepsilon^r_i\right]}f(t)\ge f\left(\tau^f_{i-1}\right)+a(c_{2,i-1})-K_1$$

- if $i\ge1$ and $f\left(\tau^f_i\right)=f\left(\tau^f_{i-1}\right)+b(c_{2,i-1})$ (i.e. the bound is crossed at the bid)

$$\max_{t\in\left[\tau^f_i-\varepsilon^l_i,\ \tau^f_i+\varepsilon^r_i\right]}f(t)\le f\left(\tau^f_{i-1}\right)+a(c_{2,i-1})-K_1$$

- if $i=0$, $\max_{t\in\left[\tau^f_0,\tau^f_0+\varepsilon^r_0\right]}|f(t)-f(0)|<K_1.$

Choose $\varepsilon^\tau=\min\left\{\min_{i\in1,\dots,L^f_T}\left\{\varepsilon^l_i;\varepsilon^r_i\right\};\ \varepsilon^r_0;\ \frac{1}{3}K_\tau;\ \frac{1}{2}\frac{\tilde{\varepsilon}K_\tau}{4T}\right\}$ and define

$$K_2=\min\left\{\begin{array}{c}\min_{i=1,\dots,L^f_T}\min\left\{\begin{array}{l}\inf_{t\in\left[\tau^f_{i-1}+\varepsilon^\tau,\ \tau^f_i-\varepsilon^\tau\right]}\left|f(t)-f\left(\tau^f_{i-1}\right)-a\left(c^f_{2,i-1}\right)\right|;\\ \inf_{t\in\left[\tau^f_{i-1}+\varepsilon^\tau,\ \tau^f_i-\varepsilon^\tau\right]}\left|f(t)-f\left(\tau^f_{i-1}\right)-b\left(c^f_{2,i-1}\right)\right|\end{array}\right\};\\ \min\left\{\begin{array}{l}\inf_{t\in\left[\tau^f_{L^f_T}+\varepsilon^\tau,T\right]}\left|f(t)-f\left(\tau^f_{L^f_T}\right)-a\left(c^f_{2,L^f_T}\right)\right|;\\ \inf_{t\in\left[\tau^f_{L^f_T}+\varepsilon^\tau,T\right]}\left|f(t)-f\left(\tau^f_{L^f_T}\right)-b\left(c^f_{2,L^f_T}\right)\right|\end{array}\right\}\end{array}\right\},$$

$$\kappa_i := \begin{cases} \inf_{t\in\left[\tau_i^f-\varepsilon^\tau,\ \tau_i^f+\varepsilon^\tau\right]}\left(f\left(\tau_i^f\right)+a\left(c_{2,i}^f\right)-f(t)\right), & \text{if } f\left(\tau_i^f\right)=f\left(\tau_{i-1}^f\right)+a\left(c_{2,i-1}\right) \\ \inf_{t\in\left[\tau_i^f-\varepsilon^\tau,\ \tau_i^f+\varepsilon^\tau\right]}\left(f(t)-f\left(\tau_i^f\right)-b\left(c_{2,i}^f\right)\right), & \text{if } f\left(\tau_i^f\right)=f\left(\tau_{i-1}^f\right)+b\left(c_{2,i-1}\right) \end{cases},$$

$$K_3 := \min_{i=1,\ldots,L_T^f}\kappa_i,$$

$$\chi_i := \begin{cases} \sup_{t\in\left[\tau_i^f,\ \tau_i^f+\varepsilon^\tau\right]}\left(f(t)-f\left(\tau_{i-1}^f\right)-a\left(c_{2,i-1}^f\right)\right), & \text{if } f\left(\tau_i^f\right)=f\left(\tau_{i-1}^f\right)+a\left(c_{2,i-1}\right) \\ \sup_{t\in\left[\tau_i^f,\ \tau_i^f+\varepsilon^\tau\right]}\left(f\left(\tau_{i-1}^f\right)+b\left(c_{2,i-1}^f\right)-f(t)\right), & \text{if } f\left(\tau_i^f\right)=f\left(\tau_{i-1}^f\right)+b\left(c_{2,i-1}\right) \end{cases}$$

$$K_4 := \min_{i=1,\ldots,L_T^f}\chi_i.$$

Note that $K_j > 0$, $j = 1, .., 4$, given our choice of ε^τ and since $f \in \mathcal{C}$.

Define the constants

$$\begin{aligned} M &:= \max_{[0,T]} e^{f(t)}\frac{2(1+\delta q)}{1-\delta},\ m := \max\left\{\max_{[0,T]} e^{-f(t)};\ 1\right\}, \\ K &:= \frac{1}{4}\min\left\{K_4;\ K_2;\ \frac{K_1}{2Mm+1};\ \log 2;\ \frac{1}{Mm};\ \frac{K_3}{2Mm+1}\right\}, \\ C^i &:= \sum_{j=1}^{i}\max\left\{(2Mm),1\right\}^j + \max\left\{(2Mm)^i,1\right\},\ i=0,...,L_T^f, \\ C &:= C^{L_T^f+1} \end{aligned}$$

Let $\varepsilon^f = \frac{1}{4}\min\{\tilde{\varepsilon}, 1\}\min\left\{\frac{K}{C+1}, 1\right\}$.

Since $f^n \to f \in \mathcal{C}$ in Skorokhod topology, therefore in uniform topology over $[0, T]$, there exists a $\bar{N}$ such that, for any $n > \bar{N}$, $\sup_{t\in[0,T]}|f^n(t) - f(t)| < \varepsilon^f$. For such n, conditions (61)-(63) are indeed satisfied as we are about to show. To prove this we are left to show by induction that, for all i,

$$\left|\tau_i^f - \tau_i^n\right| < \varepsilon^\tau,\ \tau_i^n > \tau_{i-1}^f + \varepsilon^\tau, \tag{64}$$

$$c_{2,i}^f = c_{2,i}^n := c_{2,i}^{f^n} \tag{65}$$

$$\left|f\left(\tau_i^f\right) - \log g^n\left(\tau_i^n\right)\right| \leq C^i\varepsilon^f \tag{66}$$

$$\left|g^n\left(\tau_i^n\right) - e^{f\left(\tau_i^f\right)}\right| < 2MC^i\varepsilon^f \tag{67}$$

and that $L_T^n = L_T^f$.

Consider $i=0$. We have $\tau_0^f=\tau_0^n=0$, $c_{2,0}^f=c_{2,0}^n=1$, and

$$
\begin{aligned}
\left|f\left(\tau_0^f\right)-\log g^n\left(\tau_0^n\right)\right| &\le C^0\varepsilon^f \text{ (since } \log g^n\left(\tau_0^n\right)=f^n\left(\tau_0^n\right)\text{)},\\
\left|g^n\left(\tau_0^n\right)-e^{f\left(\tau_0^f\right)}\right| &\le M\left|e^{\log g^n\left(\tau_0^n\right)-f\left(\tau_0^f\right)}-1\right|\le 2M\varepsilon^f.
\end{aligned}
$$

To show that $\tau_1^n>\tau_0^f+\varepsilon^\tau$ note that, for $t\in[0,\varepsilon^\tau]$

$$
\begin{aligned}
a(1)+f^n(0)-f^n(t) &\ge a(1)-2\varepsilon^f-K_1\ge a(1)-\frac{9}{8}K_1\\
&=\log\frac{q}{q-\delta}-\frac{9}{8}K_1\ge\frac{7}{8}\log\frac{q}{q-\delta}
\end{aligned}
$$

due to the choice of ε^f and K_1. Similarly

$$
f^n(t)-b(1)-f^n(0)\ge\frac{7}{8}\log\frac{q}{q+\delta}\qquad\text{for } t\in[0,\varepsilon^\tau].
$$

Thus, $\tau_1^n>\tau_0^f+\varepsilon^\tau$.

Suppose the assumptions of induction hold for $i-1$.

- To show that $\tau_i^n>\tau_i^f-\varepsilon^\tau$, observe that, for $t\in\left[\tau_{i-1}^f+\varepsilon^\tau,\tau_i^f-\varepsilon^\tau\right]$,

$$
\begin{aligned}
a\left(c_{2,i-1}\right)+f^n\left(\tau_{i-1}^n\right)-f^n(t) &\ge K_2-2\varepsilon^f\ge\frac{7}{8}K_2\\
f^n(t)-b\left(\tau_{i-1}^n\right)-f^n\left(\tau_{i-1}^n\right) &\ge\frac{7}{8}K_2
\end{aligned}
$$

  due to the choice of ε^f and K_2. Thus $\tau_i^n>\tau_i^f-\varepsilon^\tau$.

- Next, to show that $\tau_i^n\in\left[\tau_i^f-\varepsilon^\tau,\tau_i^f+\varepsilon^\tau\right]$, we need two observations. First, note that if $f\left(\tau_i^f\right)=f\left(\tau_{i-1}^f\right)+a\left(c_{2,i-1}\right)$ (i.e. the bound is crossed at ask), then

$$
\begin{aligned}
&\inf_{t\in\left[\tau_i^f-\varepsilon^\tau,\ \tau_i^f+\varepsilon^\tau\right]}\left[a\left(c_{2,i-1}\right)+\log g^n\left(\tau_{i-1}^n\right)-f^n(t)\right]\\
&\le\inf_{t\in\left[\tau_i^f-\varepsilon^\tau,\ \tau_i^f+\varepsilon^\tau\right]}\left[a\left(c_{2,i-1}\right)+f\left(\tau_{i-1}^f\right)+C^{i-1}\varepsilon^f-f(t)+\varepsilon^f\right]\\
&\le C\varepsilon^f+\varepsilon^f-K_4\le\frac{1}{4}K-K_4\le-\frac{15}{16}K_4<0,
\end{aligned}
$$

  hence f^n crosses its upper boundary in this interval whenever f crossed at ask.
  Second, note that if $f\left(\tau_i^f\right)=f\left(\tau_{i-1}^f\right)+b\left(c_{2,i-1}\right)$ (i.e. the bound is crossed at bid), we have

$$
\inf_{t\in\left[\tau_i^f-\varepsilon^\tau,\ \tau_i^f+\varepsilon^\tau\right]}\left[f^n(t)-b\left(c_{2,i-1}\right)-\log g^n\left(\tau_{i-1}^n\right)\right]\le-\frac{15}{16}K_4<0.
$$

  Hence, f^n crosses its lower boundary in this interval whenever f crossed at bid.

As a consequence, f^n crosses one of its bounds over this interval i.e. $\tau_i^n \in \left[\tau_i^f - \varepsilon^\tau, \tau_i^f + \varepsilon^\tau\right]$, and obviously $\left|\tau_i^f - \tau_i^n\right| < \varepsilon^\tau$.

- To show that $c_{2,i}^f = c_{2,i}^n$ i.e. that f^n crosses at ask (bid) whenever f does so, we need two observations. First, note that if $f\left(\tau_i^f\right) = f\left(\tau_{i-1}^f\right) + a\left(c_{2,i-1}\right)$, then for $t \in \left[\tau_i^f - \varepsilon^\tau, \tau_i^f + \varepsilon^\tau\right]$

$$\begin{aligned} f^n(t) - b\left(c_{2,i-1}\right) - \log g^n\left(\tau_{i-1}^n\right) &\geq f(t) - 2\varepsilon^f - b\left(c_{2,i-1}\right) - C^{i-1}\varepsilon^f - f\left(\tau_{i-1}^f\right) \\ &\geq f(t) - 2\varepsilon^f + \log\frac{q+\delta}{q-\delta} - a\left(c_{2,i-1}\right) - C^{i-1}\varepsilon^f - f\left(\tau_{i-1}^f\right) \\ &\geq \frac{13}{16}K_1 > 0. \end{aligned}$$

That is the i-th trade at time τ_i^n cannot happen at bid in this case.
Second, note that if $f\left(\tau_i^f\right) = f\left(\tau_{i-1}^f\right) + b\left(c_{2,i-1}\right)$, then for $t \in \left[\tau_i^f - \varepsilon^\tau, \tau_i^f + \varepsilon^\tau\right]$

$$a\left(c_{2,i-1}\right) + \log g^n\left(\tau_{i-1}^n\right) - f^n(t) \geq \frac{13}{16}K_1 > 0,$$

hence the i-th trade at time τ_i^n cannot happen at ask in this case.
Therefore, $c_{2,i}^f = c_{2,i}^n$.

- To verify the induction statements (66) and (67) we need to consider two cases. First, if $f^n\left(\tau_i^n\right) > a\left(c_{2,i-1}\right) + \log g^n\left(\tau_{i-1}^n\right)$ (i.e. f^n crossed its bound at ask), then

$$f^n\left(\tau_i^n\right) > a\left(c_{2,i-1}\right) + \log g^n\left(\tau_{i-1}^n\right) \geq f\left(\tau_i^f\right) - 2C^{i-1}\varepsilon^f.$$

Moreover, since f^n cannot have jumps larger than $2\varepsilon^f$ (since otherwise its distance from f would become more than ε^f), and since f^n should be below its upper bound before crossing it, we have

$$f^n\left(\tau_i^n\right) \leq a\left(c_{2,i-1}\right) + \log g^n\left(\tau_{i-1}^n\right) + 2\varepsilon^f \leq f\left(\tau_i^f\right) + 2\left(C^{i-1}+1\right)\varepsilon^f.$$

Therefore, $\left|f^n\left(\tau_i^n\right) - f\left(\tau_i^n\right)\right| \leq 2\left(C^{i-1}+1\right)\varepsilon^f$. This implies that

$$\begin{aligned} &\left|g^n\left(\tau_i^n\right) - e^{f\left(\tau_i^f\right)}\right| = \frac{1}{c_{2,i}}\left|\left[(1-q)\left(e^{f^n\left(\tau_i^n\right)} - e^{f\left(\tau_i^f\right)}\right) + q\left(g^n\left(\tau_{i-1}^n\right) - e^{f\left(\tau_{i-1}^f\right)}\right)c_{2,i-1}\right]\right| \\ &\leq \frac{1}{c_{2,i}}\left[(1-q)\,e^{f\left(\tau_i^f\right)}\left|e^{f^n\left(\tau_i^n\right) - f\left(\tau_i^f\right)} - 1\right| + qe^{f\left(\tau_i^f\right)}\left|e^{\log g^n\left(\tau_{i-1}^n\right) - f\left(\tau_{i-1}^f\right)} - 1\right|c_{2,i-1}\right] \\ &\leq \frac{2e^{f\left(\tau_i^f\right)}}{c_{2,i}}\left[(1-q)\left|f^n\left(\tau_i^n\right) - f\left(\tau_i^f\right)\right| + q\left|\log g^n\left(\tau_{i-1}^n\right) - f\left(\tau_{i-1}^f\right)\right|c_{2,i-1}\right] \\ &\leq \frac{4e^{f\left(\tau_i^f\right)}}{1-\delta}\left[1+q\delta\right]\varepsilon^f\left(C^{i-1}+1\right) \leq 2M\varepsilon^f\left(C^{i-1}+1\right) \end{aligned}$$

where the third inequality is due to the fact that $\left|e^x - 1\right| < 2|x|$ whenever $|x| \leq \varepsilon^f\left(C^{i-1}+1\right) <$

$\log 2$. Hence $\left|g^n\left(\tau_i^n\right)-e^{f\left(\tau_i^f\right)}\right|<2MC^i\varepsilon^f$ as claimed in the induction. Furthermore

$$\begin{aligned}\left|\log g^n\left(\tau_i^n\right)-f\left(\tau_i^f\right)\right| &= \left|\log\left(1+\frac{g^n\left(\tau_i^n\right)-e^{f\left(\tau_i^f\right)}}{e^{f\left(\tau_i^f\right)}}\right)\right|\leq 2\left|\frac{g^n\left(\tau_i^n\right)-e^{f\left(\tau_i^f\right)}}{e^{f\left(\tau_i^f\right)}}\right|\\ &\leq 2Mm\left(C^{i-1}+1\right)\varepsilon^f\leq C^i\varepsilon^f\end{aligned}$$

since $|\log(1+x)|\leq 2|x|$ for $|x|\leq 2Mm\left(C^{i-1}+1\right)\varepsilon^f\leq C^i\varepsilon^f<1/2$.
Second, if $f^n\left(\tau_i^n\right)<b\left(c_{2,i-1}\right)+\log g^n\left(\tau_{i-1}^n\right)$ (i.e. f^n crossed its bound at bid), then $f^n\left(\tau_i^n\right)\leq f\left(\tau_i^f\right)+2C^{i-1}\varepsilon^f$. Moreover, we have that $f^n\left(\tau_i^n\right)\geq f\left(\tau_i^f\right)-2\left(C^{i-1}+1\right)\varepsilon^f$. Therefore, $\left|f^n\left(\tau_i^n\right)-f\left(\tau_i^n\right)\right|\leq 2\left(C^{i-1}+1\right)\varepsilon^f$. This implies that

$$\left|g^n\left(\tau_i^n\right)-e^{f\left(\tau_i^f\right)}\right|\leq 2M\varepsilon^f\left(C^{i-1}+1\right),$$

therefore $\left|g^n\left(\tau_i^n\right)-e^{f\left(\tau_i^f\right)}\right|<2MC^i\varepsilon^f$ as claimed in the induction. Hence, as before,

$$\left|\log g^n\left(\tau_i^n\right)-f\left(\tau_i^f\right)\right|\leq C^i\varepsilon^f.$$

- To show that f^n does not cross more than once one of its boundaries on the interval $\left[\tau_i^f-\varepsilon^\tau,\tau_i^f+\varepsilon^\tau\right]$, i.e. $\tau_{i+1}^n>\tau_i^f+\varepsilon^\tau$, we need the following two observations.
First, if $f^n\left(\tau_i^n\right)>a\left(c_{2,i-1}\right)+\log g^n\left(\tau_{i-1}^n\right)$, this implies that (as shown above) $f\left(\tau_i^f\right)=a\left(c_{2,i-1}\right)+f\left(\tau_{i-1}^f\right)$, therefore for $t\in\left[\tau_i^f-\varepsilon^\tau,\tau_i^f+\varepsilon^\tau\right]$

$$\begin{aligned}f^n(t)-b\left(c_{2,i}\right)-\log g^n\left(\tau_i^n\right) &\geq f(t)-\varepsilon^f-b\left(c_{2,i}\right)-f\left(\tau_i^f\right)-C^i\varepsilon^f\\ &\geq f\left(\tau_{i-1}^f\right)-f\left(\tau_i^f\right)+a\left(c_{2,i-1}\right)-b\left(c_{2,i}\right)-K_1-\left(C^i+1\right)\varepsilon^f\\ &= -\log\frac{q(1-\delta)}{q+\delta}-K_1-\left(C^i+1\right)\varepsilon^f\geq\frac{15}{16}K_1>0.\end{aligned}$$

Hence, if $\tau_{i+1}^n\in\left[\tau_i^f-\varepsilon^\tau,\tau_i^f+\varepsilon^\tau\right]$, it cannot happen at bid. On the other hand,

$$\begin{aligned}a\left(c_{2,i}\right)+\log g^n\left(\tau_i^n\right)-f^n(t) &\geq a\left(c_{2,i}\right)+f\left(\tau_i^f\right)-f(t)-\varepsilon^f-C^i\varepsilon^f\\ &\geq K_3-\left(C^i+1\right)\varepsilon^f\geq\frac{15}{16}K_3>0.\end{aligned}$$

Hence, if $\tau_{i+1}^n\in\left[\tau_i^f-\varepsilon^\tau,\tau_i^f+\varepsilon^\tau\right]$, it cannot happen at ask either. Thus, $\tau_{i+1}^n\notin\left[\tau_i^f-\varepsilon^\tau,\tau_i^f+\varepsilon^\tau\right]$.
Second, if $f^n\left(\tau_i^n\right)<b\left(c_{2,i-1}\right)+\log g^n\left(\tau_{i-1}^n\right)$, this implies that (as shown above) $f\left(\tau_i^f\right)=$

$b\left(c_{2,i-1}\right)+f\left(\tau_{i-1}^{f}\right)$, therefore for $t\in\left[\tau_{i}^{f}-\varepsilon^{\tau},\tau_{i}^{f}+\varepsilon^{\tau}\right]$

$$\begin{aligned}
a\left(c_{2,i}\right)+\log g^{n}\left(\tau_{i}^{n}\right)-f^{n}\left(t\right) &\geq a\left(c_{2,i}\right)+f\left(\tau_{i}^{f}\right)-f\left(t\right)-\varepsilon^{f}-C^{i}\varepsilon^{f}\\
&\geq a\left(c_{2,i}\right)+f\left(\tau_{i-1}^{f}\right)+b\left(c_{2,i-1}\right)-f\left(t\right)-\left(C^{i}+1\right)\varepsilon^{f}\\
&\geq \log\frac{q\left(1+\delta\right)}{q-\delta}-K_{1}-\left(C^{i}+1\right)\varepsilon^{f}\geq\frac{15}{16}K_{1}>0.
\end{aligned}$$

Also

$$f^{n}\left(t\right)-b\left(c_{2,i}\right)-\log g^{n}\left(\tau_{i}^{n}\right)\geq f\left(t\right)-b\left(c_{2,i}\right)-f\left(\tau_{i}^{f}\right)-\varepsilon^{f}-C^{i}\varepsilon^{f}\geq\frac{15}{16}K_{3}>0.$$

Therefore, $\tau_{i+1}^{n}\notin\left[\tau_{i}^{f}-\varepsilon^{\tau},\tau_{i}^{f}+\varepsilon^{\tau}\right]$ in this case too.

- Thus, by the principle of mathematical induction, the statements (64)-(65) hold for $i=1,...,L_{T}^{f}$.

To complete the proof of the Lemma, we need to establish that $L_{T}^{f}=L_{T}^{n}$ for $n>\bar{N}$. By the above we have that $L_{t}^{f}=L_{t}^{n}$ for any $t\leq\tau_{L_{T}^{f}}^{f}+\varepsilon^{\tau}$, thus the only thing left to show is that $\tau_{L_{T}^{f}+1}^{n}\notin\left[\tau_{L_{T}^{f}}^{f}+\varepsilon^{\tau},T\right)$. Observe that, for $t\in\left[\tau_{L_{T}^{f}}^{f}+\varepsilon^{\tau},T\right)$, and $i=L_{T}^{f}$

$$\begin{aligned}
&a\left(c_{2,i}\right)+f^{n}\left(\tau_{i}^{n}\right)-f^{n}\left(t\right)\geq K_{2}-2\varepsilon^{f}\geq\frac{7}{8}K_{2},\\
&f^{n}\left(t\right)-b\left(c_{2,i}\right)-f^{n}\left(\tau_{i}^{n}\right)\geq\frac{7}{8}K_{2}.
\end{aligned}$$

Thus $\tau_{L_{T}^{f}+1}^{n}\geq T$. ■

Proof of Lemma 6. We first derive the conditional expectation of time between two consecutive trades. In particular we will prove that the following conjecture holds for all $i>j\geq1$

$$\mathbb{E}\left[\tau_{i}-\tau_{i-1}-\mu_{\tau}\mid\mathcal{F}_{\tau_{i-j}}^{W}\right]=\begin{cases}S(q+\delta)(1+\delta)\left(\frac{q^{2}-\delta^{2}}{q(1-\delta^{2})}\right)^{j-1}, & \text{if }\phi_{i-j}=\frac{q}{q-\delta}\\-S(q-\delta)(1-\delta)\left(\frac{q^{2}-\delta^{2}}{q(1-\delta^{2})}\right)^{j-1}, & \text{if }\phi_{i-j}=\frac{q}{q+\delta}\end{cases}\tag{68}$$

where μ_{τ} is defined in Lemma 6 and

$$S:=\frac{1}{\sigma^{2}(q+\delta^{2})}\left[\frac{q^{2}-\delta^{2}}{q(1-\delta^{2})}\log\frac{q-\delta}{q+\delta}+\log\frac{1+\delta}{1-\delta}\right].$$

The proof is by induction on j. First, consider $j=1$. We have, for any $t\in\mathbb{R}_{+}$,

$$\mathbb{E}\left[\tau_{i}\wedge t-\tau_{i-1}\wedge t\mid\mathcal{F}_{\tau_{i-1}}^{W}\right]=-\frac{2}{\sigma}\mathbb{E}\left[W_{\tau_{i}\wedge t}-W_{\tau_{i-1}\wedge t}-\frac{\sigma}{2}\left(\tau_{i}\wedge t-\tau_{i-1}\wedge t\right)\mid\mathcal{F}_{\tau_{i-1}}^{W}\right]$$

by Theorem 7.29 Kallenberg (2002). Observe that the left hand side is monotonically increasing in t and the right hand side takes values in the interval $\left[-\frac{2}{\sigma^{2}}a\left(\frac{q}{\phi_{i-1}}+1-q\right),-\frac{2}{\sigma^{2}}b\left(\frac{q}{\phi_{i-1}}+1-q\right)\right]$

and is therefore bounded. Thus, taking the limit as $t \to \infty$ and applying the Monotone Convergence Theorem (to the left hand side) and the Dominated Convergence Theorem (to the right hand side) yields:

$$
\begin{aligned}
&\mathbb{E}\left[\tau_i - \tau_{i-1} \mid \mathcal{F}^W_{\tau_{i-1}}\right] = -\tfrac{2}{\sigma}\mathbb{E}\left[W_{\tau_i} - W_{\tau_{i-1}} - \tfrac{\sigma}{2}\left(\tau_i - \tau_{i-1}\right) \mid \mathcal{F}^W_{\tau_{i-1}}\right] \\
&= -\tfrac{2}{\sigma^2}\mathbb{E}\left[a\left(\tfrac{q}{\phi_{i-1}} + 1 - q\right)\mathbf{1}_{\left\{\phi_i = \frac{q}{q-\delta}\right\}} + b\left(\tfrac{q}{\phi_{i-1}} + 1 - q\right)\mathbf{1}_{\left\{\phi_i = \frac{q}{q+\delta}\right\}} \mid \mathcal{F}^W_{\tau_{i-1}}\right] \\
&= -\tfrac{2}{\sigma^2}\left[a\left(\tfrac{q}{\phi_{i-1}} + 1 - q\right)\mathbb{P}\left[\phi_i = \tfrac{q}{q-\delta} \mid \phi_{i-1}\right] + b\left(\tfrac{q}{\phi_{i-1}} + 1 - q\right)\mathbb{P}\left[\phi_i = \tfrac{q}{q+\delta} \mid \phi_{i-1}\right]\right] \\
&= \mu_\tau + \begin{cases} S(q+\delta)(1+\delta), & \text{if } \phi_{i-1} = \frac{q}{q-\delta} \\ -S(q-\delta)(1-\delta), & \text{if } \phi_{i-1} = \frac{q}{q+\delta} \end{cases},
\end{aligned}
$$

where the second equality is due to the definition of τ_i in Theorem 7, the third one is due to the fact that ϕ is Markov, and the last one is obtained via direct calculations by employing the conditional probabilities of Corollary 1.

Next, suppose that the statement of the induction is true for $j = n$. Let $j = n + 1$ and observe that for any $i > j$ we have

$$
\begin{aligned}
&\mathbb{E}\left[\tau_i - \tau_{i-1} \mid \mathcal{F}^W_{\tau_{i-(n+1)}}\right] = \mathbb{E}\left[\mathbb{E}\left[\tau_i - \tau_{i-1} \mid \mathcal{F}^W_{\tau_{i-n}}\right] \mid \mathcal{F}^W_{\tau_{i-n-1}}\right] \\
&= \mu_\tau + S\left(\tfrac{q^2-\delta^2}{q(1-\delta^2)}\right)^{n-1}\mathbb{E}\left[(q+\delta)(1+\delta)\mathbf{1}_{\left\{\phi_{i-n} = \frac{q}{q-\delta}\right\}} - (q-\delta)(1-\delta)\mathbf{1}_{\left\{\phi_{i-n} = \frac{q}{q+\delta}\right\}} \mid \phi_{i-n-1}\right],
\end{aligned}
$$

where the last equality is due to the assumption of the induction and the fact that ϕ is Markov. Using as before the conditional probabilities given in Corollary 1, direct calculation proves that the conjecture holds.

Next, note that $\tau_n - \tau_{n-1} - \mu_\tau$ is a $\mathcal{L}^2$ mixingale since[24]

$$
\begin{aligned}
\left\|\mathbb{E}\left[\tau_i - \tau_{i-1} - \mu_\tau \mid \mathcal{F}^W_{\tau_{i-n}}\right]\right\|_2 &= S\left(\tfrac{q^2-\delta^2}{q(1-\delta^2)}\right)^{n-1}\left[\mathbb{E}\left((q+\delta)(1+\delta)\mathbf{1}_{\left\{\phi_{i-n} = \frac{q}{q-\delta}\right\}}\right.\right. \\
&\qquad \left.\left. -(q-\delta)(1-\delta)\mathbf{1}_{\left\{\phi_{i-n} = \frac{q}{q+\delta}\right\}}\right)^2\right]^{\frac{1}{2}} \\
&\leq S\sqrt{2}(q+\delta)(1+\delta)\left(\tfrac{q^2-\delta^2}{q(1-\delta^2)}\right)^{n-1},
\end{aligned}
$$

where the first equality is due to the result above and the inequality is due the fact that $(a+b)^2 \leq 2(a^2+b^2)$. Moreover, let $c_n = c = S\sqrt{2}(q+\delta)(1+\delta)$ and $\Psi(n) = \left(\frac{q^2-\delta^2}{q(1-\delta^2)}\right)^{n-1}$ and observe that $\Psi(n) = o(\log^{-2}(n))$. Hence, by Corollary 1 of de Jong (1995), we have that

$$
\frac{\tau_n}{n} \underset{n\to\infty}{\longrightarrow} \mu_\tau \ a.s.
$$

The second statement of the Lemma is proved by contradiction. Fix an $\omega \in \{\omega \in \Omega : \lim_{n\to\infty} \frac{\tau_n(\omega)}{n} = \mu_\tau\}$. Suppose that for this ω there exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to\infty} \frac{L^p_{t_i}(\omega)}{t_i} = K \neq \frac{1}{\mu_\tau}$, where K can take infinity as a value.

[24]For a definition of mixingales see e.g. de Jong (1995).

If $K = +\infty$, then for any $M \in \mathbb{R}_+$ there exists an $N \in \mathbb{N}$ such that for any $n \geq N$ we have

$$\frac{L^p_{t_n}(\omega)}{t_n} > M \Leftrightarrow \tau_{\lfloor Mt_n \rfloor}(\omega) < t_n \Leftrightarrow \frac{\tau_{\lfloor Mt_n \rfloor}(\omega)}{\lfloor Mt_n \rfloor} < \frac{t_n}{\lfloor Mt_n \rfloor}$$

where the operator $\lfloor \cdot \rfloor$ returns the largest integer smaller than its argument. Taking the limit yields that $\lim_{n\to\infty} \frac{\tau_{\lfloor Mt_n \rfloor}(\omega)}{\lfloor Mt_n \rfloor} \leq \frac{1}{M}$ for any $M \in \mathbb{R}_+$ and, therefore, is equal to zero, which contradicts the choice of ω as, evidently, $\mu_\tau \neq 0$.

If $K < +\infty$, we have two possibilities: either $K < \frac{1}{\mu_\tau}$ or $K > \frac{1}{\mu_\tau}$. We will consider only the first case as the second one can be dealt with in similar manner.

Fix an $\varepsilon = \frac{1}{4}\left(\frac{1}{\mu_\tau} - K\right)$. As $\lim_{i\to\infty} \frac{L^p_{t_i}(\omega)}{t_i} = K$ there exists an $N \in \mathbb{N}$ such that, for any $n \geq N$, we have $\frac{L^p_{t_n}(\omega)}{t_n} - K < \varepsilon$. Observe that we have

$$L^p_{t_n}(\omega) < t_n(\varepsilon + K) \Leftrightarrow \tau_{\lfloor t_n(\varepsilon+K)\rfloor+1}(\omega) > t_n \Leftrightarrow \frac{\tau_{\lfloor t_n(\varepsilon+K)\rfloor+1}(\omega)}{\lfloor t_n(\varepsilon+K)\rfloor+1} > \frac{t_n}{\lfloor t_n(\varepsilon+K)\rfloor+1}.$$

Taking the limit yields that $\lim_{n\to\infty} \frac{\tau_{\lfloor t_n(\varepsilon+K)\rfloor+1}(\omega)}{\lfloor t_n(\varepsilon+K)\rfloor+1} \geq \frac{1}{\varepsilon+K} > \mu_\tau$ due to the choice of ε, which contradicts the choice of ω .

Thus, for any $\omega \in \{\omega \in \Omega : \lim_{n\to\infty} \frac{\tau_n(\omega)}{n} = \mu_\tau\}$ we have $\lim_{t\to\infty} \frac{L_t(\omega)}{t} = \frac{1}{\mu_\tau}$. ■

A.2 Proof of Proposition 9

To prove Proposition 9 we first need to establish a few auxiliary results.

Definition 2 (Most Recent Common Ancestor) *Consider y^n_i defined in Equations (28)-(31). We define the most recent common ancestor of y^n_i and y^n_j, $\mathcal{A}\left(y^n_i, y^n_j\right)$, recursively as follows*

$$\mathcal{A}\left(y^n_i, y^n_i\right) = i,$$
$$\mathcal{A}\left(y^n_i, y^n_j\right) = \mathcal{A}\left(y^n_j, y^n_i\right) = \mathbf{1}_{\{i>j\}} \sum_{k=0}^{i-1} \zeta_{i-1,k}\mathcal{A}\left(y^n_j, y^n_k\right) + \mathbf{1}_{\{i<j\}} \sum_{k=0}^{j-1} \zeta_{j-1,k}\mathcal{A}\left(y^n_k, y^n_i\right).$$

Lemma 7 *Suppose $q < 1$, then for any $i \geq j \geq k \geq 0$, and any $a \in \left(\max\left\{\sqrt{q}, \sqrt{\frac{q^4}{4} + 2q} - \frac{q^2}{2}\right\}, 1\right)$ we have*

$$\bar{\mathbb{P}}\left(\mathcal{A}\left(y^n_i, y^n_j\right) = k\right) \leq ca^{2j-i-k}.$$

where $c = a^{-2M} > 1$, and M is the smallest nonnegative integer m such that $q^s + q^{s+(1-2s)m}\left(1 - q^{s+1}\right) < 1$,[25] *with $s < 1/2$ being the solution of $a = q^s$.*

[25] Such M exists since

$$\lim_{m\to\infty} q^s + q^{s+(1-2s)m}\left(1 - q^{s+1}\right) = q^s < 1.$$

Moreover, note that for all $m \geq M$

$$q^s + q^{s+(1-2s)m}\left(1 - q^{s+1}\right) < 1$$

Proof. The proof is by induction on the maximum of i and j.
Suppose that $\max\{i,j\} = n \leq M$. Then the assumption of mathematical indiction holds since

$$\bar{\mathbb{P}}\left(\mathcal{A}\left(y_i^n, y_j^n\right) = k\right) \leq 1 \leq ca^{2j} \leq ca^{2j-i-k}$$

due to the definition of c.
Suppose that for $\max\{i,j\} = m \geq M$ the assumption of induction holds. Consider $\max\{i,j\} = i = m+1$, then we have the following four cases.

1. $k \neq 0$, $k \neq j$.

$$\begin{aligned}\bar{\mathbb{P}}\left(\mathcal{A}\left(y_{m+1}^n, y_j^n\right) = k\right) &= \bar{\mathbb{P}}\left(\sum_{l=0}^{m} \zeta_{m,l}\mathcal{A}\left(y_l^n, y_j^n\right) = k\right) = \sum_{l=0}^{m} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = k\right)\bar{\mathbb{P}}\left(\zeta_{m,l} = 1\right)\\ &= \sum_{l=k}^{m} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = k\right)\bar{\mathbb{P}}\left(\zeta_{m,l} = 1\right) = \sum_{l=k}^{m} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = k\right)(1-q)\,q^{m-l}\end{aligned}$$

where the third equality is due to the fact that the common ancestor $\mathcal{A}\left(y_l^n, y_j^n\right) \leq \min\{l, j\}$, and the fourth follows from the definition of ζ in Equation (31). Then

$$\begin{aligned}&\bar{\mathbb{P}}\left(\mathcal{A}\left(y_{m+1}^n, y_j^n\right) = k\right) = \sum_{l=k}^{j-1} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = k\right)(1-q)\,q^{m-l} + \sum_{l=j+1}^{m} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = k\right)(1-q)\,q^{m-l}\\ &\leq (1-q)\,c\left(\sum_{l=k}^{j} a^{2l-j-k}q^{m-l} + \sum_{l=j+1}^{m} a^{2j-l-k}q^{m-l}\right)\\ &= \frac{(1-q)\,c}{(1-aq)\left(a^2-q\right)}\left\{a^{2j-m-k}\left(a^2-q\right) + \underbrace{a^{k-j}q^{m+1-k}\left(aq-1\right)}_{\text{negative for } a<1} + \underbrace{\left[q^{m-j}a^{j-k}\left(-aq^2+2q-a^2\right)\right]}_{\text{negative for } a>\sqrt{\frac{q^4}{4}+2q}-\frac{q^2}{2}}\right\}\\ &\leq \frac{(1-q)\,c}{(1-aq)\left(a^2-q\right)}\left\{a^{2j-m-k}\left(a^2-q\right)\right\} = \frac{(1-q)\,c}{(1-aq)}a^{2j-m-k} \leq ca^{2j-m-k-1}\end{aligned}$$

where the first equality comes from $k \neq j$, the first inequality follows from the principle of mathematical induction and the last inequality follows from the conditions on a.[26]

2. $k \neq 0$, $k = j$.

$$\bar{\mathbb{P}}\left(\mathcal{A}\left(y_{m+1}^n, y_j^n\right) = k\right) = \bar{\mathbb{P}}\left(\sum_{l=0}^{m} \zeta_{m,l}\mathcal{A}\left(y_l^n, y_j^n\right) = k\right) = \sum_{l=0}^{m} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = k\right)\bar{\mathbb{P}}\left(\zeta_{m,l} = 1\right)$$

$$= \sum_{l=j}^{m} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = k\right)\bar{\mathbb{P}}\left(\zeta_{m,l} = 1\right) = \sum_{l=j+1}^{m} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = k\right)(1-q)\,q^{m-l} + (1-q)\,q^{m-k}$$

---

since the left hand side is monotone in m.

[26] Since $\sqrt{\frac{q^4}{4} + 2q - \frac{q^2}{2}} < 1$ there exists an a satisfying the conditions in Lemma 7.

where the last equality is due to $\bar{\mathbb{P}}\left(\mathcal{A}\left(y_k^n, y_k^n\right) = k\right) = 1$. By induction we have

$$(1-q)\left[\sum_{l=j+1}^{m} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = k\right) q^{m-l} + q^{m-k}\right] \le c(1-q)\left\{\sum_{l=j+1}^{m} a^{2j-l-k} q^{m-l} + q^{m-k}\right\}$$

$$= c(1-q)\, a^{k-m-1}\left\{\frac{a - a^{-k+m+2} q^{m-k+1}}{(1-aq)}\right\} \le \frac{a(1-q)}{(1-aq)} c a^{k-m-1} \le c a^{k-m-1}$$

3. $k = 0, k \neq j$.

$$\bar{\mathbb{P}}\left(\mathcal{A}\left(y_{m+1}^n, y_j^n\right) = 0\right) = \bar{\mathbb{P}}\left(\sum_{l=0}^{m} \zeta_{m,l} \mathcal{A}\left(y_l^n, y_j^n\right) = 0\right) = \sum_{l=0}^{m} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = 0\right) \bar{\mathbb{P}}\left(\zeta_{m,l} = 1\right)$$

$$= \sum_{l=1}^{j-1} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = 0\right)(1-q) q^{m-l} + \sum_{l=j+1}^{m} \bar{\mathbb{P}}\left(\mathcal{A}\left(y_l^n, y_j^n\right) = 0\right)(1-q) q^{m-l}$$

$$+ \bar{\mathbb{P}}\left(\mathcal{A}\left(y_0^n, y_j^n\right) = 0\right) q^m$$

where the last term follows from the definition of ζ in Equation (31). By induction

$$\bar{\mathbb{P}}\left(\mathcal{A}\left(y_{m+1}^n, y_j^n\right) = 0\right) \le c(1-q)\left\{\sum_{l=1}^{j-1} a^{2l-j} q^{m-l} + \sum_{l=j+1}^{m} a^{2j-l} q^{m-l}\right\} + c a^{-j} q^m$$

$$= c(1-q)\left\{q^{m-j} a^j \underbrace{\frac{q^j a^{2-2j}(aq-1) - aq^2 - a^2 + 2q}{(a^2-q)(1-aq)}}_{\text{negative for: } 1>a>\sqrt{\frac{q^4}{4}+2q}-\frac{q^2}{2}} + \frac{a^{2j-m}}{1-aq}\right\} + c a^{-j} q^m$$

$$\le c\left\{(1-q)\frac{a^{2j-m}}{1-aq} + a^{-j} q^m\right\} = c a^{2j-m-1}\left\{(1-q)\frac{a}{1-aq} + a^{m-3j+1} q^m\right\}$$

The proof of this case is completed by showing that the last term above is smaller than 1, and this is the case if

$$a + a^{m-3j+1} q^m - a^{m-3j+2} q^{m+1} < 1.$$

Note that $a = q^s$, for some $s < 1/2$, hence

$$a + a^{m-3j+1} q^m - a^{m-3j+2} q^{m+1} < q^s + q^{s+(1-2s)m}\left(1 - q^{s+1}\right) < 1$$

where the first inequality is due to $m \ge j$, and the second is due to $m \ge M$.

4. $k = j = 0$.

$$\bar{\mathbb{P}}\left(\mathcal{A}\left(y_{m+1}^n, y_0^n\right) = 0\right) = 1 < c a^{-m-1} = a^{-2M-m-1}.$$

Hence by the principle of mathematical induction the statement of the Lemma holds for all m. ■

Lemma 8 *Let ψ denote*

$$\psi_i^n = \overline{\mathbb{E}}\left[\bar{Y}_T^n | \mathcal{H}_i^n\right] - \overline{\mathbb{E}}\left[\bar{Y}_T^n | \mathcal{H}_{i-1}^n\right] \tag{69}$$

where $\overline{\mathbb{E}}$ denotes expectations taken with respect to the measure $\bar{\mathbb{P}}$ and $\mathcal{H}_i^n := \mathcal{F}_{\theta_i^n}^n$ with $\mathcal{F}_t^n := \sigma\left\{\bar{Y}_{s\leq t}^n\right\}$. Denote the variance of ψ with $\left(\sigma_{n,i}^{\psi}\right)^2 = \overline{\mathbb{E}}\left[(\psi_i^n)^2\right]$. The following holds for any $t \geq 0$:

1. $\lim_{n\to\infty} \max\limits_{i\leq N_t^n} \sigma_{n,i}^{\psi} = 0$.

2. *The set*

$$K^{\psi} := \left\{ \frac{(\psi_i^n)^2}{\left(\sigma_{n,i}^{\psi}\right)^2},\ n \in \mathbb{N},\ i \leq N_t^n \right\}$$

   is uniformly integrable.

3. *For any $k > 0$*

$$\lim_{n\to\infty} \bar{\mathbb{P}}\left[\max_{i\leq N_t^n} |\psi_i^n| > k\right] = 0$$

Proof of Lemma 8. We prove the assertions of the Lemma in the same order as stated.

1. Note that from Lemma 4 and the definition of $\bar{Y}$ we have

$$\overline{\mathbb{E}}\left[\bar{Y}_T^n | \mathcal{H}_i^n\right] = (1-q)\left[y_i^n + \mu\left(T - \theta_i^n\right)\right] + q\overline{\mathbb{E}}\left[\bar{Y}_T^n | \mathcal{H}_{i-1}^n\right],$$

   from which it follows that

$$\frac{\psi_i^n}{1-q} = \left[y_i^n - \mu\theta_i^n - D_0\right] - \sum_{j=1}^{i-1} \psi_j^n, \tag{70}$$

   therefore

$$\frac{\left(\sigma_{n,i}^{\psi}\right)^2}{(1-q)^2} = \sigma^2\theta_i^n - \sum_{j=1}^{i-1}\left(\sigma_{n,j}^{\psi}\right)^2.$$

   Thus, by induction

$$\frac{\left(\sigma_{n,i}^{\psi}\right)^2}{(1-q)^2} = \sigma^2 \sum_{j=1}^{i}\left\{\left[q\left(2-q\right)\right]^{i-j}\Delta_{j,j-1}^n\right\} \tag{71}$$

   where $\Delta_{j,j-1}^n := \theta_j^n - \theta_{j-1}^n$ and note that $q\left(2-q\right) < 1$ for all $q \in [0,1]$.
   Fix any $\omega \in \Omega$, than from Ω_3 it follows that there exists a $k\left(\omega\right)$ such that $n\Delta_{j,j-1}^n < 2\log\left(j\right)$ for any $j \geq k$. Moreover, from Ω_4 we have that $\lim_{n\to\infty} \frac{N_T^n}{n} = \lambda T$. Therefore from

equation (71) we have

$$\begin{aligned}
\left(\sigma_{n,i}^{\psi}\right)^{2} &= (1-q)^{2}\sigma^{2}\sum_{j=1}^{i}\left\{[q(2-q)]^{i-j}\,\Delta_{j,j-1}^{n}\right\}\\
&< (1-q)^{2}\sigma^{2}\left\{\sum_{j=1}^{k-1}\left\{[q(2-q)]^{i-j}\,\Delta_{j,j-1}^{n}\right\}+\frac{2}{n}\sum_{j=k}^{i}[q(2-q)]^{i-j}\log(j)\right\}\\
&< (1-q)^{2}\sigma^{2}\left\{\sum_{j=1}^{k-1}\Delta_{j,j-1}^{n}+\frac{2}{n}\log(i)\sum_{j=k}^{i}[q(2-q)]^{i-j}\right\}\\
&= (1-q)^{2}\sigma^{2}\left\{\sum_{j=1}^{k-1}\Delta_{j,j-1}^{n}+\frac{2}{n}\log(i)\sum_{j=k}^{i}[q(2-q)]^{i-j}\right\}\\
&= (1-q)^{2}\sigma^{2}\left\{\sum_{j=1}^{k-1}\Delta_{j,j-1}^{n}+\frac{2}{n}\log(i)\,\frac{1-[q(2-q)]^{i-k+1}}{1-[q(2-q)]}\right\}\\
&< (1-q)^{2}\sigma^{2}\left\{\sum_{j=1}^{k-1}\Delta_{j,j-1}^{n}+\frac{2}{n}\log(N_{T}^{n})\,\frac{1}{1-[q(2-q)]}\right\}\\
&= (1-q)^{2}\sigma^{2}\left\{\sum_{j=1}^{k-1}\frac{\Delta_{j,j-1}}{n}+\frac{2}{n}\log(N_{T}^{n})\,\frac{1}{1-[q(2-q)]}\right\}
\end{aligned}$$

$$\begin{aligned}
\therefore\quad \left(\sigma_{n,i}^{\psi}\right)^{2} &= (1-q)^{2}\sigma^{2}\sum_{j=1}^{i}\left\{[q(2-q)]^{i-j}\,\Delta_{j,j-1}^{n}\right\}\\
&\le (1-q)^{2}\sigma^{2}\left\{\sum_{j=1}^{k-1}\frac{\Delta_{j,j-1}}{n}+\frac{2}{n}\log(N_{T}^{n})\,\frac{1}{1-[q(2-q)]}\right\}
\end{aligned}\tag{72}$$

Note that the right hand side does not depend on i and its limit as $n\to\infty$ is zero. Thus $\lim_{n\to\infty}\max\limits_{i\le N_t^n}\sigma_{n,i}^{\psi}=0$.

2. Consider $\left(\kappa_{n,i}^{\psi}\right)^{4} := \overline{\mathbb{E}}\left[(\psi_i^n)^4\right]$ and note that from Equation (70) at arrival i and $i-1$, it follows that

$$\left(\kappa_{n,i}^{\psi}\right)^{4} \le (1-q)^{4}\,\overline{\mathbb{E}}\left[\left(y_i^n-y_{i-1}^n-\mu\Delta_{i,i-1}^n\right)^4\right]$$

$$
\begin{aligned}
&= \sigma^4 (1-q)^4 \sum_{j=0}^{i-1} \bar{\mathbb{P}} \left(\mathcal{A}\left(y_i^n, y_{i-1}^n\right) = j\right) \bar{\mathbb{E}} \left[\left(\sqrt{\Delta_{i,j}^n} \eta_{i,j} - \sqrt{\Delta_{i-1,j}^n} \eta_{i-1,j} \right)^4 \right] \\
&= 3\sigma^4 (1-q)^4 \sum_{j=0}^{i-1} \bar{\mathbb{P}} \left(\mathcal{A}\left(y_i^n, y_{i-1}^n\right) = j\right) \left[\theta_i^n - \theta_{i-1}^n + 2 \sum_{k=j+1}^{i-1} \left(\theta_k^n - \theta_{k-1}^n\right) \right]^2 \\
&\le 12\sigma^4 (1-q)^4 \sum_{j=0}^{i-1} \bar{\mathbb{P}} \left(\mathcal{A}\left(y_i^n, y_{i-1}^n\right) = j\right) (i-j-1) \sum_{k=j+1}^{i} \left(\theta_k^n - \theta_{k-1}^n\right)^2 \\
&\le 12\sigma^4 (1-q)^4 c \sum_{j=0}^{i-1} \sum_{k=j+1}^{i} a^{i-j-2} (i-j) \left(\theta_k^n - \theta_{k-1}^n\right)^2 \\
&= 12\sigma^4 (1-q)^4 ca^{-2} \sum_{k=1}^{i} \left(\theta_k^n - \theta_{k-1}^n\right)^2 \sum_{j=0}^{k-1} a^{i-j} (i-k-(j-k)) \\
&\le 12\sigma^4 (1-q)^4 ca^{-2} \sum_{k=1}^{i} \left(\theta_k^n - \theta_{k-1}^n\right)^2 a^{i-k} \left[(i-k) \sum_{j=0}^{k} a^{k-j} - \sum_{j=0}^{k} a^{k-j} (j-k) \right] \\
&= 12\sigma^4 (1-q)^4 ca^{-2} \sum_{k=1}^{i} \left(\theta_k^n - \theta_{k-1}^n\right)^2 a^{i-k} \left[(i-k) \sum_{l=0}^{k} a^l + \sum_{l=0}^{k} a^l l \right]
\end{aligned}
$$

where the first equality follows form the definition of y_i^n, the second from the fact that $\eta_{i,j}$ is and iid standard normal and the definition of $\Delta_{i,j}^n$, while the second inequality comes for the observation that $\left(\frac{1}{n}\sum_{i=1}^n a_i\right)^2 \le \frac{1}{n}\sum_{i=1}^n a_i^2$ and the third inequality follows from Lemma 7, the third equality is simply a change in the summations order, the third inequality comes from having added one nonnegative element to the sum over j, and finally the last equality is obtained setting $l = k - j$.

Note that for any $b \in (a, 1)$ there exists a constant c_1 such that

$$
\left(x + \frac{a}{1-a}\right) \left(\frac{a}{b}\right)^x \le c_1 \qquad \forall x \in [0, \infty).
$$

Therefore

$$
\left(\kappa_{n,i}^{\psi}\right)^4 \le \frac{12\sigma^4 (1-q)^4 ca^{-2}}{1-a} \sum_{k=1}^{i} \left(\theta_k^n - \theta_{k-1}^n\right)^2 a^{i-k} \left[(i-k) + \frac{a}{1-a} \right] \le c_2 \sum_{k=1}^{i} \left(\theta_k^n - \theta_{k-1}^n\right)^2 b^{i-k} \tag{73}
$$

where $c_2 := 12\sigma^4 (1-q)^4 ca^{-2} c_1 / (1-a)$.

Now to prove that the set K^{ψ} is almost surely uniformly integrable we need to show

$$
\sup_{n,i} \frac{\left(\kappa_{n,i}^{\psi}\right)^4}{\left(\sigma_{n,i}^{\psi}\right)^4} < \infty.
$$

From Equations (71) and (73) we have

$$\frac{\left(\kappa^{\psi}_{n,i}\right)^4}{\left(\sigma^{\psi}_{n,i}\right)^4} \le c_3 \frac{\sum_{j=1}^{i}\left(\theta^n_j - \theta^n_{j-1}\right)^2 b^{i-j}}{\left\{\sum_{j=1}^{i}\left[q\left(2-q\right)\right]^{i-j}\left(\theta^n_j - \theta^n_{j-1}\right)\right\}^2} \le c_3 \frac{\sum_{j=1}^{i} b^{i-j}\left(\theta^n_j - \theta^n_{j-1}\right)^2}{\sum_{j=1}^{i} b_1^{i-j}\left(\theta^n_j - \theta^n_{j-1}\right)^2} \tag{74}$$

$$= c_3 \frac{\sum_{j=1}^{i} b^{i-j}\left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2}{\sum_{j=1}^{i} b_1^{i-j}\left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2} \tag{75}$$

where $c_3 := c_2/\left(1-q\right)^4\sigma^4$ and $b_1 := \left[q\left(2-q\right)\right]^2$ and the last equality follows from Equation (27).

Now consider a random variable X_i with distribution given by

$$\bar{\mathbb{P}}\left(X_i = \frac{i-j}{i}\right) = \frac{\left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2}{\sum_{j=1}^{i}\left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2}.$$

Then, for any $s \in [0,1]$, we have the cumulative distribution function

$$F_i\left(s\right) = \bar{\mathbb{P}}\left(X_i \le s\right) = \frac{\sum_{j=1}^{\lfloor si \rfloor}\left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2}{\sum_{j=1}^{i}\left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2}$$

and, given the regularity condition Ω_1, this cdf is such that $\lim_{i\to\infty} F_i\left(s\right) = s$. Therefore, from Theorem III.1.2 of Shiryaev (1996) we have that X_i weakly converges to a uniform random variable, i.e. $X_i \xrightarrow[i\to\infty]{w} X \sim \mathcal{U}\left(0,1\right)$, and in particular

$$\lim_{i\to\infty} \bar{\mathbb{E}}\left[e^{-kX_i}\right] = \frac{1-e^{-k}}{k} \quad \forall k > 0.$$

Now notice that, using the definition of X_i, the ratio in Equation (75) can be rewritten as

$$\frac{\sum_{j=1}^{i} b^{i-j}\left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2}{\sum_{j=1}^{i} b_1^{i-j}\left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2} = \frac{\bar{\mathbb{E}}\left[e^{iX_i \ln b}\right]}{\bar{\mathbb{E}}\left[e^{iX_i \ln b_1}\right]}$$

where $\ln b$ and $\ln b_1$ are both negative. Therefore, to establish uniform integrability of K^{ψ} it is sufficient to show uniform convergence of $\bar{\mathbb{E}}\left[e^{-kX_i}\right]$ in k. To do so consider the following class of equicontinuous, uniformly bounded functions $\boldsymbol{S} := \left\{s : \mathbb{R}_+ \to \mathbb{R}_+ : s\left(x\right) = \frac{e^{-kx}}{k},\ k \in [1,\infty)\right\}$. Then, from Theorem III.8.3 of Shiryaev (1996), we have that

$$\lim_{i\to\infty} \sup_{k\in[1,\infty)} \left|\bar{\mathbb{E}}\left[\frac{e^{-kX_i}}{k}\right] - \left(1 - e^{-k}\right)\right| = 0.$$

Therefore, for any $\epsilon \in (0,1)$, there exists a $\bar{\imath}$ such that for any $i \ge \bar{\imath}$

$$\frac{\sum_{j=1}^{i} b^{i-j}\left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2}{\sum_{j=1}^{i} b_1^{i-j}\left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2} \le \frac{\ln b}{\ln b_1}\left(1+\epsilon\right).$$

Thus

$$\sup_{n,i} \frac{\left(\kappa_{n,i}^{\psi}\right)^4}{\left(\sigma_{n,i}^{\psi}\right)^4} \leq c_3 \sup_i \frac{\sum_{j=1}^{i} b^{i-j} \left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2}{\sum_{j=1}^{i} b_1^{i-j} \left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2}$$

$$\leq c_3 \max \left\{ \frac{\ln b}{\ln b_1} (1+\epsilon), \max_{i \leq \bar{\imath}} \frac{\sum_{j=1}^{i} b^{i-j} \left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2}{\sum_{j=1}^{i} b_1^{i-j} \left(\bar{\gamma}_j - \bar{\gamma}_{j-1}\right)^2} \right\} < \infty,$$

implying that K^{ψ} is uniformly integrable.

3. To prove that for any $k > 0$, $\lim_{n\to\infty} \bar{\mathbb{P}} \left[\max_{i \leq N_t^n} |\psi_i^n| > k \right] = 0$, first observe that from Equation (73), due to the regularity condition Ω_3, there exists an $\bar{\imath} < \infty$ such that

$$\bar{\mathbb{E}} \left[(\psi_i^n)^4 \right] \equiv \left(\kappa_{n,i}^{\psi}\right)^4 \leq c_2 \sum_{k=1}^{i} \left(\theta_k^n - \theta_{k-1}^n\right)^2 b^{i-k} = \frac{c_2}{n^2} \sum_{k=1}^{i} \left(\bar{\gamma}_k - \bar{\gamma}_{k-1}\right)^2 b^{i-k}$$
$$\leq \frac{c_2}{n^2} \left[\sum_{k=1}^{\bar{\imath}} \left(\bar{\gamma}_k - \bar{\gamma}_{k-1}\right)^2 b^{i-k} + 4 \sum_{k=\bar{\imath}+1}^{i} (\ln k)^2 b^{i-k} \right] \leq \frac{c_2}{n^2} \left[\sum_{k=1}^{\bar{\imath}} \left(\bar{\gamma}_k - \bar{\gamma}_{k-1}\right)^2 + \frac{4 (\ln i)^2}{1-b} \right]. \tag{76}$$

Consider a random variable $\chi(n)$ given by $\chi(n) := \arg\max_{j \leq n} \left| \psi_j^n \right|$. Then

$$\bar{\mathbb{P}} \left[\max_{i \leq N_t^n} |\psi_i^n| > k \right] = \sum_{i \leq N_t^n} \bar{\mathbb{P}} \left[\psi_i^n > k | \chi(N_t^n) = i \right] \bar{\mathbb{P}} \left(\chi(N_t^n) = i \right)$$
$$\leq \sum_{i \leq N_t^n} \frac{\bar{\mathbb{E}} \left[(\psi_i^n)^2 | \chi(N_t^n) = i \right] \bar{\mathbb{P}} \left(\chi(N_t^n) = i \right)}{k^2} = \sum_{i \leq N_t^n} \frac{\bar{\mathbb{E}} \left[(\psi_i^n)^2 \mathbf{1}_{\{\chi(N_t^n) = i\}} \right]}{k^2}$$
$$\leq \sum_{i \leq N_t^n} \frac{\left\{ \bar{\mathbb{E}} \left[(\psi_i^n)^4 \right] \bar{\mathbb{P}} \left(\chi(N_t^n) = i \right) \right\}^{1/2}}{k^2} \leq \frac{c_2^{1/2}}{k^2} \left[\frac{1}{N_t^n} \sum_{k=1}^{\bar{\imath}} \left(\bar{\gamma}_k - \bar{\gamma}_{k-1}\right)^2 + \frac{4 (\ln N_t^n)^2}{N_t^n (1-b)} \right]^{1/2} \frac{N_t^n}{n}$$

Where the first inequality is the Chebyshev's inequality, the second inequality is the Cauchy-Buniakovsky inequality, the third inequality comes from Equation (76) and the observation that $\sum_{i=1}^{n} x_i^{1/2} \leq \sqrt{n} \left(\sum_{i=1}^{n} x_i\right)^{1/2}$.

Hence from Ω_2, Ω_4 and the fact that $\lim_{x\to\infty} (\ln x)^2 / x = 0$, we finally have

$$\lim_{n\to\infty} \bar{\mathbb{P}} \left[\max_{i \leq N_t^n} |\psi_i^n| > k \right] = 0$$

as claimed.

■

Lemma 9 *Consider y_i^n defined in defined in Equations (28)-(31). Then*

$$\bar{\mathbb{P}}\left[y_i^n \le y | y_{i-k}^n, ..., y_0^n\right] = (1-q)\sum_{j=1}^{i-k} q^{i-k-j}\bar{\mathbb{P}}\left[y_j^n + \varepsilon_{i,j}^n \le y | y_j^n\right] + q^{i-k}\bar{\mathbb{P}}\left[y_0^n + \varepsilon_{i,0}^n \le y | y_0^n\right].$$

Proof. The proof is by the principle of mathematical induction. For $k = 1$ the statement is trivially true given the definition of y_i^n. Suppose the statement holds for $k = m$, that is

$$\bar{\mathbb{P}}\left[y_i^n \le y | y_{i-m}^n, ..., y_0^n\right] = (1-q)\sum_{j=1}^{i-m} q^{i-m-j}\bar{\mathbb{P}}\left[y_j^n + \varepsilon_{i,j}^n \le y | y_j^n\right] + q^{i-m}\bar{\mathbb{P}}\left[y_0^n + \varepsilon_{i,0}^n \le y | y_0^n\right].$$

We than have that for $k = m + 1$

$$\begin{aligned}
&\bar{\mathbb{P}}\left[y_i^n \le y | y_{i-m-1}^n, ..., y_0^n\right] = \bar{\mathbb{E}}\left[\bar{\mathbb{E}}\left[\mathbf{1}_{\{y_i^n \le y\}}\middle| y_{i-m}^n, ..., y_0^n\right]\middle| y_{i-m-1}^n, ..., y_0^n\right] \\
&= \bar{\mathbb{E}}\left[(1-q)\sum_{j=1}^{i-m} q^{i-m-j}\bar{\mathbb{P}}\left[y_j^n + \varepsilon_{i,j}^n \le y | y_j^n\right] + q^{i-m}\bar{\mathbb{P}}\left[y_0^n + \varepsilon_{i,0}^n \le y \middle| y_0^n\right]\middle| y_{i-m-1}^n, ..., y_0^n\right] \\
&= (1-q)\,\bar{\mathbb{E}}\left[\bar{\mathbb{E}}\left[\mathbf{1}_{\{y_{i-m}^n + \varepsilon_{i,i-m}^n \le y\}}\middle| y_{i-m}^n\right]\middle| y_{i-m-1}^n, ..., y_0^n\right] \qquad (77) \\
&+ (1-q)\sum_{j=1}^{i-m-1} q^{i-m-j}\bar{\mathbb{P}}\left[y_j^n + \varepsilon_{i,j}^n \le y \middle| y_j^n\right] + q^{i-m}\bar{\mathbb{P}}\left[y_0^n + \varepsilon_{i,0}^n \le y \middle| y_0^n\right].
\end{aligned}$$

Note that

$$\begin{aligned}
&\bar{\mathbb{E}}\left[\bar{\mathbb{E}}\left[\mathbf{1}_{\{y_{i-m}^n + \varepsilon_{i,i-m}^n \le y\}}\middle| y_{i-m}^n\right]\middle| y_{i-m-1}^n, ..., y_0^n\right] = \bar{\mathbb{E}}\left[\bar{\mathbb{E}}\left[\mathbf{1}_{\{y_{i-m}^n + \varepsilon_{i,i-m}^n \le y\}}\middle| y_{i-m}^n, ..., y_0^n\right]\middle| y_{i-m-1}^n, ..., y_0^n\right] \\
&= \bar{\mathbb{E}}\left[\bar{\mathbb{E}}\left[\mathbf{1}_{\{y_{i-m}^n \le y - \varepsilon_{i,i-m}^n\}}\middle| y_{i-m-1}^n, ..., y_0^n, \varepsilon_{i,i-m}^n\right]\middle| y_{i-m-1}^n, ..., y_0^n\right] \\
&= (1-q)\sum_{j=1}^{i-m-1} q^{i-m-1-j}\bar{\mathbb{P}}\left[y_j^n + \varepsilon_{i-m,j}^n + \varepsilon_{i,i-m}^n \le y | y_j^n\right] + q^{i-m-1}\bar{\mathbb{P}}\left[y_0^n + \varepsilon_{i,0}^n \le y | y_0^n\right] \\
&= (1-q)\sum_{j=1}^{i-m-1} q^{i-m-1-j}\bar{\mathbb{P}}\left[y_j^n + \varepsilon_{i,j}^n \le y | y_j^n\right] + q^{i-m-1}\bar{\mathbb{P}}\left[y_0^n + \varepsilon_{i,0}^n \le y | y_0^n\right],
\end{aligned}$$

where the first two equalities come from the independence of ε, the third comes form the statement of the induction with $i = i - m$, $m = 1$ and $y = y - \varepsilon$, and the last comes from the Gaussianity and independence of ε. Combining this result with equation (77) yields

$$\bar{\mathbb{P}}\left[y_i^n \le y | y_{i-m-1}^n, ..., y_0^n\right] = (1-q)\sum_{j=1}^{i-m-1} q^{i-m-1-j}\bar{\mathbb{P}}\left[y_j^n + \varepsilon_{i,j}^n \le y | y_j^n\right] + q^{i-m-1}\bar{\mathbb{P}}\left[y_0^n + \varepsilon_{i,0}^n \le y | y_0^n\right].$$

QED. ■

We now can establish Proposition 9.

Proof of Proposition 9. The proof of the proposition proceeds as follows. First, we establish the tightness of $\bar{Y}^n$ in Skorokhod topology, and the fact that the limiting process is a continuos local martingale. Second, we establish the joint tightness of the processes $\bar{Y}^n$, $e^{\bar{Y}^n}$, and $\bar{\theta}^n$, where $\bar{\theta}^n_t := \theta^n_{N^n_t}$. Third, we identify the limiting processes.

To establish tightness of $\bar{Y}^n$, consider $M^n_t := \sum_{i=1}^{N^n_t} \psi^n_i$ where ψ^n_i is defined in Lemma 8. Due to Lemma 8 and Theorem 2.4 of McLeish (1977), the sequence of processes $(M^n, \mathcal{F}^n)$ is tight[27] in Stone (1963) topology for $q \in (0,1)$. Since by definition Stone's topology is equivalent to Skorokhod topology on $\mathbb{D}([0,T])$ (the space of *cádlág* processes in the $[0,T]$ interval), $(M^n, \mathcal{F}^n)$ is tight in Skorokhod topology as well.

Moreover, since by Lemma 8

$$\lim_{n\to\infty} \bar{\mathbb{P}}\left(\max_{t\le K} |\Delta M^n_t| > k\right) = \lim_{n\to\infty} \bar{\mathbb{P}}\left(\max_{i\le N^n_k} |\psi^n_i| > k\right) = 0, \tag{78}$$

and the sequence M^n is tight we have that it is C-tight, that is all limit points of the sequence $\{\mathcal{L}(M^n)\}$ are laws of continuos processes (see Proposition VI.3.26 Jacod and Shiryaev (2003)).

Furthermore, consider any convergent subsequence of M^n, M^{n_k}, then by Equation (78) and the Borel-Cantelli Lemma there exist a further subsequence, denoted for simplicity by n, such that $\max_{t\le N^n_T} |\Delta M^n_t| \to 0$ a.s. $\bar{\mathbb{P}}$. Therefore, it there exists m and c such that for all $n \ge m$, $|\Delta M^n_t| \le c \ \forall t \in [0,T]$. Hence, the limit process of $(M^n, \mathcal{F}^n)$ is a local martingale (see Proposition IX.1.17 Jacod and Shiryaev (2003)). Finally, since the choice of the converging subsequence was arbitrary, we have that all the limits of $(M^n, \mathcal{F}^n)$ are continuos local martingales.

Note that from Lemma 4 and the definition of $\bar{Y}$ we have

$$\overline{\mathbb{E}}\left[\bar{Y}^n_T | \mathcal{H}^n_i\right] = (1-q)\left[y^n_i + \mu\left(T - \theta^n_i\right)\right] + q\overline{\mathbb{E}}\left[\bar{Y}^n_T | \mathcal{H}^n_{i-1}\right].$$

Since

$$\sum_{i=1}^{N^n_t} \psi^n_i = \overline{\mathbb{E}}\left[\bar{Y}^n_T | \mathcal{H}^n_{N^n_t}\right] - \overline{\mathbb{E}}\left[\bar{Y}^n_T | \mathcal{H}^n_0\right] = \overline{\mathbb{E}}\left[\bar{Y}^n_T | \mathcal{H}^n_{N^n_t}\right] - \mu T$$

it follows that

$$\bar{Y}^n_t - \mu T = \sum_{i=1}^{N^n_t - 1} \psi^n_i + \frac{q}{1-q}\psi^n_{N^n_t} = M^n_t + \frac{2q-1}{1-q}\psi^n_{N^n_t}.$$

Due to condition (78), and the fact that M^n_t is C-tight and its limit is a continuous local martingale, we have from Lemma VI.3.31 and Proposition VI.3.17 of Jacod and Shiryaev (2003), that $\bar{Y}^n_t$ is also C-tight and its limit is a continuous local martingale.

We now turn to the joint tightness of $\bar{Y}^n$, $e^{\bar{Y}^n}$ and $\bar{\theta}^n$. Observe that $\bar{\theta}^n$, given the definition of θ^n, is such that

$$\bar{\theta}^n_t = \sum_{i=1}^{N^n_t} \frac{\bar{\gamma}_i - \bar{\gamma}_{i-1}}{n} \to t \quad \text{for all } t \in [0,T], \ \omega \in \Omega.$$

[27] See page 309 of Kallenberg (2002).

Moreover,

$$\sum_{i=1}^{N_t^n} \frac{\left(\bar{\gamma}_i - \bar{\gamma}_{i-1}\right)^2}{n^2} \to 0 \quad \text{for all } t \in [0,T]\,,\, \omega \in \Omega.$$

Thus, by Theorem VI.2.2.15 of Jacod and Shiryaev (2003) we have that $\bar{\theta}^n \to \bar{\theta}$ in Skorokhod topology where $\bar{\theta}_t = t$.

Consider now any convergent subsequence of $\bar{Y}^n$, without loss of generality let it be denoted by n, then it follows form the tightness result that there exist a continuous local martingale $\bar{Y}$ such that $\mathcal{L}\left(\bar{Y}^n\right) \to \mathcal{L}\left(\bar{Y}\right)$. Since $g : \mathbb{D}\left([0,T]\right) \to \mathbb{D}\left([0,T]\right) : g\left(x_t\right) = e^{x_t}$ is a continuos map for continuos processes in Skorokhod topology,[28] by Proposition VI.3.8.II of Jacod and Shiryaev (2003), we have that $\mathcal{L}\left(e^{\bar{Y}^n}\right) \to \mathcal{L}\left(e^{\bar{Y}}\right)$. By Corollary VI.3.33b of Jacod and Shiryaev (2003), we then have that the sequence $\left(\bar{Y}^n, e^{\bar{Y}^n}, \bar{\theta}^n\right)$ is C-tight, and for any converging subsequence $\bar{Y}^n$, $\mathcal{L}\left(\bar{Y}^n, e^{\bar{Y}^n}, \bar{\theta}^n\right) \to \mathcal{L}\left(\bar{Y}, e^{\bar{Y}}, \bar{\theta}\right)$.

We can finally identify the limiting processes. Form the above it is clear that the only part left to identify is $\bar{Y}$. Assume, wlog, that $\bar{Y}^n$ is a converging subsequence. Theorem III.8.1 of Shiryaev (1996) states that we can define a probability space, and a sequence of processes X^n, such that $X^n \to X$ almost surely in Skorokhod topology, and such that $\mathcal{L}\left(\bar{Y}^n\right) = \mathcal{L}\left(X^n\right)$ and $\mathcal{L}\left(\bar{Y}\right) = \mathcal{L}\left(X\right)$. Therefore, since we are only interested in the distribution of $\bar{Y}$ we can assume, wlog, that $\bar{Y}^n$ converges to $\bar{Y}$ not only in law, but also almost surely in Skorokhod topology.

By Lemma 9, we have that for any $t > s > 0$

$$\begin{aligned}
&\bar{\mathbb{P}}\left[\bar{Y}_t^n \le y | \mathcal{F}_s^{\bar{Y}^n}\right] = \bar{\mathbb{P}}\left[y_{N_t^n}^n \le y - \mu\left(T - \bar{\theta}_t^n\right) \Big| \, y_{N_s^n}^n, ..., y_0^n\right] \\
&= (1-q) \sum_{j=1}^{N_s^n} q^{N_s^n - j} \bar{\mathbb{P}}\left[\bar{y}_j^n + \sigma \sqrt{\Delta_{N_t^n,j}^n} \eta_{N_t^n,j} \le y | \bar{y}_j^n\right] + q^{N_s^n} \bar{\mathbb{P}}\left[\bar{y}_0^n + \sigma \sqrt{\Delta_{N_t^n,0}^n} \eta_{N_t^n,0} \le y | \bar{y}_0^n\right] \\
&= (1-q) \sum_{j=1}^{N_s^n} q^{N_s^n - j} \int_{-\infty}^{\frac{y - \bar{y}_j^n}{\sigma \sqrt{\Delta_{N_t^n,j}^n}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx + q^{N_s^n} \bar{\mathbb{P}}\left[\bar{y}_0^n + \sigma \sqrt{\Delta_{N_t^n,0}^n} \eta_{N_t^n,0} \le y | \bar{y}_0^n\right] \\
&= \left(1 - q^{N_s^n}\right) \int_{-\infty}^{\frac{y - \bar{Y}_s^n}{\sigma \sqrt{\Delta_{N_t^n,N_s^n}^n}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx + (1-q) \sum_{j=1}^{N_s^n - 1} q^{N_s^n - j} \int_{\frac{y - \bar{y}_{N_s^n}^n}{\sigma \sqrt{\Delta_{N_t^n,N_s^n}^n}}}^{\frac{y - \bar{y}_j^n}{\sigma \sqrt{\Delta_{N_t^n,j}^n}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\
&+ q^{N_s^n} \bar{\mathbb{P}}\left[\bar{y}_0^n + \sigma \sqrt{\Delta_{N_t^n,0}^n} \eta_{N_t^n,0} \le y | \bar{y}_0^n\right],
\end{aligned}$$

where the second equality comes from the definition of ε and $\bar{y}$, the third equality comes from the fact that η is an independent standard Gaussian. Note that, as n goes to infinity, the last term vanishes and

$$\left(1 - q^{N_s^n}\right) \int_{-\infty}^{\frac{y - \bar{Y}_s^n}{\sigma \sqrt{\Delta_{N_t^n,N_s^n}^n}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \underset{n\to\infty}{\longrightarrow} \int_{-\infty}^{\frac{y - \bar{Y}_s}{\sigma\sqrt{t-s}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \tag{79}$$

[28] Since the Skorohod topology becomes uniform for continuous processes, see e.g. Proposition VI.1.17b of Jacod and Shiryaev (2003).

due to the almost sure convergence of $\bar{Y}^n$ and Ω_4 and Ω_5.

Note also that

$$\begin{aligned}
&\sum_{j=1}^{N_s^n-1} q^{N_s^n-j} \int_{\frac{y-\bar{y}^n_{N_s^n}}{\sigma\sqrt{\Delta^n_{N_t^n,N_s^n}}}}^{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N_t^n,j}}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \\
&= \sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N_s^n-1} q^{N_s^n-j} \int_{\frac{y-\bar{y}^n_{N_s^n}}{\sigma\sqrt{\Delta^n_{N_t^n,N_s^n}}}}^{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N_t^n,j}}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx + \sum_{j=1}^{N^n_{s-\frac{1}{\sqrt{n}}}-1} q^{N_s^n-j} \int_{\frac{y-\bar{y}^n_{N_s^n}}{\sigma\sqrt{\Delta^n_{N_t^n,N_s^n}}}}^{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N_t^n,j}}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \\
&\leq \sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N_s^n-1} q^{N_s^n-j} \int_{\frac{y-\bar{y}^n_{N_s^n}}{\sigma\sqrt{\Delta^n_{N_t^n,N_s^n}}}}^{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N_t^n,j}}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx + \frac{q^{\bar{\Lambda}_{sn-\sqrt{n}}}}{1-q},
\end{aligned}$$

where the last term goes to zero, as n goes to infinity, due to Ω_4 and the first term above can be rewritten as

$$\begin{aligned}
&\sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N_s^n-1} q^{N_s^n-j} \int_{\frac{y-\bar{y}^n_{N_s^n}}{\sigma\sqrt{\Delta^n_{N_t^n,N_s^n}}}}^{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N_t^n,j}}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \\
&= \sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N_s^n-1} q^{N_s^n-j} \int_{\frac{y-\bar{Y}_s}{\sigma\sqrt{\Delta^n_{N_t^n,N_s^n}}}}^{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N_t^n,N_s^n}}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx + \sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N_s^n-1} q^{N_s^n-j} \int_{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N_t^n,N_s^n}}}}^{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N_t^n,j}}}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx. \qquad (80)
\end{aligned}$$

To show that the above vanishes in the limit, fix an ω and consider any κ_1, $\kappa_2 > 0$. Notice that by the continuity of $\bar{Y}$, there exists a $\kappa_3 \in (0, s)$ such that $\left|\bar{Y}_s - \bar{Y}_u\right| \leq \kappa_1$ for all $u \in [s - \kappa_3, s]$.

Observe that, for n big enough and $j \in \left[N^n_{s-\frac{1}{\sqrt{n}}}, N_s^n - 1\right]$, we have

$$\bar{y}_j^n = \bar{Y}_u^n, \quad u \in [s - \kappa_3, s]$$

and, since $\bar{Y}^n$ converges almost surely in Skorokhod topology to a continuos process $\bar{Y}$, it also converges in uniform topology on compact sets,

$$\sup_{u\in[s-\kappa_3,s]} \left|\bar{Y}_u^n - \bar{Y}_u\right| \leq \kappa_2.$$

Therefore,

$$\bar{y}_j^n \in \left[\bar{Y}_s - \kappa_2 - \kappa_1, \bar{Y}_s + \kappa_2 + \kappa_1\right] \qquad \forall j \in \left[N^n_{s-\frac{1}{\sqrt{n}}}, N_s^n - 1\right].$$

To show that the first term in equation (80) vanishes, notice that the above implies that

$$\sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N^n_s-1} q^{N^n_s-j}\int_{\frac{y-\bar{Y}_s}{\sigma\sqrt{\Delta^n_{N^n_t,N^n_s}}}}^{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N^n_t,N^n_s}}}}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \leq \sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N^n_s-1} q^{N^n_s-j}\frac{(\kappa_1+\kappa_2)}{\sigma\sqrt{2\pi\Delta^n_{N^n_t,N^n_s}}} \underset{n\to\infty}{\longrightarrow} \frac{\kappa_1+\kappa_2}{\sigma(1-q)\sqrt{2\pi t}}$$

due to Ω_4 and Ω_5. Since κ_1 and κ_2 are arbitrary, we have that

$$\sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N^n_s-1} q^{N^n_s-j}\int_{\frac{y-\bar{Y}_s}{\sigma\sqrt{\Delta^n_{N^n_t,N^n_s}}}}^{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N^n_t,N^n_s}}}}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \underset{n\to\infty}{\longrightarrow} 0. \tag{81}$$

To show that the second term in equation (80) vanishes, notice that for the same κ_1, κ_2 and κ_3

$$\sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N^n_s-1} q^{N^n_s-j}\int_{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N^n_t,N^n_s}}}}^{\frac{y-\bar{y}^n_j}{\sigma\sqrt{\Delta^n_{N^n_t,j}}}}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \leq \frac{1}{\sigma\sqrt{2\pi}}\sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N^n_s-1} q^{N^n_s-j}\left|y-\bar{y}^n_j\right|\left|\frac{1}{\sqrt{\Delta^n_{N^n_t,j}}}-\frac{1}{\sqrt{\Delta^n_{N^n_t,N^n_s}}}\right|$$

$$\leq \frac{|y|+\left|\bar{Y}_s\right|+\kappa_1+\kappa_2}{\sigma\sqrt{2\pi\Delta^n_{N^n_t,N^n_{s-\frac{1}{\sqrt{n}}}}\Delta^n_{N^n_t,N^n_s}}}\frac{\left|\Delta^n_{N^n_t,N^n_s}-\Delta^n_{N^n_t,N^n_{s-\frac{1}{\sqrt{n}}}}\right|}{\sqrt{\Delta^n_{N^n_t,N^n_{s-\frac{1}{\sqrt{n}}}}}+\sqrt{\Delta^n_{N^n_t,N^n_s}}}\sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N^n_s-1} q^{N^n_s-j} \underset{n\to\infty}{\longrightarrow} 0 \tag{82}$$

Since, due to Ω_4 and Ω_5, $\Delta^n_{N^n_t,N^n_s}\to t-s$, $\Delta^n_{N^n_t,N^n_{s-\frac{1}{\sqrt{n}}}}\to t-s$, and $\sum_{j=N^n_{s-\frac{1}{\sqrt{n}}}}^{N^n_s-1} q^{N^n_s-j}\to 1/(1-q)$. Collecting the results in equations (79), (81) and (82) we have

$$\bar{\mathbb{P}}\left[\bar{Y}^n_t\leq y|\mathcal{F}^{\bar{Y}^n}_s\right] \underset{n\to\infty}{\longrightarrow} \int_{-\infty}^{\frac{y-\bar{Y}_s}{\sigma\sqrt{t-s}}}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = \bar{\mathbb{P}}\left[\bar{Y}_t\leq y|\mathcal{F}^{\bar{Y}}_s\right].$$

We also trivially have that

$$\bar{\mathbb{P}}\left[\bar{Y}^n_t\leq y\right] \underset{n\to\infty}{\longrightarrow} \int_{-\infty}^{\frac{y-\bar{Y}_0}{\sigma\sqrt{t}}}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = \bar{\mathbb{P}}\left[\bar{Y}_t\leq y\right].$$

Thus, by direct calculation we have

$$\bar{\mathbb{E}}\left[e^{\bar{Y}_t-\frac{\sigma^2}{2}t}|\mathcal{F}^{\bar{Y}}_s\right] = e^{\bar{Y}s-\frac{\sigma^2}{2}s} \quad \forall 0\leq s\leq t\leq T,$$

hence $e^{\bar{Y}_t-\frac{\sigma^2}{2}t}$ is a martingale. By Exercise 3.3.38.ii of Karatzas and Shreve (1991), we have that $\langle\bar{Y}\rangle_t=\sigma^2 t$. Therefore, by the Levy characterization of the Brownian motion (see e.g. Theorem 3.3.16 of Karatzas and Shreve (1991)), $\bar{Y}_t=\sigma W_t$ where W is a standard Brownian motion. Since the converging subsequence of $\bar{Y}^n$ was arbitrary, and since W is clearly independent of the particular realisation of $\bar{\Lambda}$, QED. ■

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