Corrigendum to "Discounted Stochastic Games, the $3M$ Property, and Stationary Markov Perfect Equilibria"

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1 Introduction

In our paper, "Discounted Stochastic Games, the $3M$ Property, and Stationary Markov Perfect Equilibria," (Fu and Page, 2022) we show that the upper Caratheodory Nash correspondence belonging to any one-shot game underlying an uncountable-compact DSG satisfying the usual assumptions where players have *convex* compact metric action sets contains *continuum-valued, minimal upper Caratheodory* Nash correspondences taking minimally essential Nash equilibrium values. We then argue that any such DSG, as a consequence of having a continuum-valued, minimal Nash correspondence, has a Nash payoff selection sub-correspondence having fixed points. While it is true that all *convex* $DSGs$ have Nash correspondences containing continuum-valued, minimal Nash correspondences, **it is not true** that, in general, these continuum-valued minimal uC Nash correspondences induce Nash payoff selection correspondences having fixed points. Continuum-valuedness is not enough. The purpose of this Corrigendum is to give a correct statement and proof of our fixed point result and restore our existence result for stationary Markov perfect equilibria. While it is true that all convex $DSGs$ have continuum-valued minimal Nash sub-correspondences (or equivalently, have $3M$ minimal Nash correspondences), it is not true that the continuum-valuedness of minimal uC Nash sub-correspondences guarantees, in general, that the induced Nash payoff selection sub-correspondences have fixed points. What is true, as we will show here, is that if the probability space of states underlying a convex DSG is nonatomic, then the uC Nash correspondence induces a weak star upper semicontinuous Nash payoff selection correspondence taking contractible values - and this is enough to restore our fixed point result for Nash payoff selection correspondences belonging to a nonatomic convex DSG. The critical property, inherently possessed by all Nash payoff selection correspondences belonging to a nonatomic convex $DSGs$ is the *K-limit property.* A Nash payoff selection correspondence has the K-limit property if and only if its graph contains all of its Komlos limits (i.e., if and only if its graph is K-closed). Thus in a nonatomic, convex DSG, it is not the $3M$ property that is critical to the existence of fixed points - and therefore, $SMPE$ - it is the K-limit property that is sufficient to guarantee that the Nash payoff selection correspondence has fixed points - and therefore, that the DSG has stationary Markov perfect equilibria.

In a DSG, the measurable selection valued Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, is given by

$$\mathcal{S}^{\infty}(\mathcal{P}_v) := \{U_{(\cdot)} \in \mathcal{L}^{\infty}_Y : U_\omega \in \mathcal{P}(\omega, v) \text{ a.e. } [\mu]\}$$

where $\mathcal{P}(\cdot, \cdot)$ is the upper Caratheodory Nash payoff correspondence given by the composition of the m-tuple of real-valued Caratheodory player payoff functions,

$$(\omega, y, x) \longrightarrow U(\omega, v, x) := (U_1(\omega, v_1, x), \ldots, U_m(\omega, v_m, x)), \tag{1}$$

with the upper Caratheodory (uC) Nash correspondence,

$$(\omega, v) \longrightarrow \mathcal{N}(\omega, v), \tag{2}$$

i.e., the Nash payoff correspondence is given by,

$$\left.\begin{array}{c}(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := U(\omega, v, \mathcal{N}(\omega, v)) \\ := \{(U_1(\omega, v_1, x), \ldots, U_m(\omega, v_m, x)) : x \in \mathcal{N}(\omega, v)\}.\end{array}\right\} \tag{3}$$

We will show that in a nonatomic convex DSG, because $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ has the *K-limit property* and takes decomposable values in $\mathcal{L}^{\infty}_{Y}$, it is weak star upper semicontinuous and takes contractible values in $\mathcal{L}^{\infty}_{Y}$.[1] As a consequence of $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ being upper semicontinuous and contractibly-valued, it is approximable, and therefore, has fixed points, implying that the DSG to which $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ belongs has stationary Markov perfect equilibria.

It is important to note that upper semicontinuity and contractible-valuedness are critical to the existence of fixed points, and therefore, critical to the existence of stationary Markov perfect equilibria. Not only do they guarantee the existence of one-shot Nash equilibria, but more importantly, they rule out the existence of circular one-shot Nash payoffs - a key pathology in the construction of counterexamples to existence (see Levy 2013 and Levy-McLennan 2015).[2] In the absence of the K-limit property and the near equivalence of K-convergence and weak star convergence, and without a nonatomic dominating probability measure and the Lyapunov machinery it makes available, it would be very difficult to construct a proof that all nonatomic, convex $DSGs$ have Nash payoff selection correspondences that are upper semicontinuous and contractibly-valued - and both are required in order to establish the existence of fixed points - and therefore the existence of $SMPE$.

Now to the details.

2 Nonatomic Convex One-Shot Games

A one-shot game (OSG), underlying a discounted stochastic game, is a *collection* of strategic form games,

$$\mathcal{G}(\Omega \times \mathcal{L}^{\infty}_{Y}) := \{\mathcal{G}(\omega, v) : (\omega, v) \in \Omega \times \mathcal{L}^{\infty}_{Y}\}, \tag{4}$$

where each (ω, v)-game in the collection is given by

$$\mathcal{G}(\omega, v) := \left\{\underbrace{\Phi_d(\omega)}_{\text{feasible actions}}, \underbrace{U_d(\omega, v_d, (\cdot, \cdot))}_{\text{payoff function}}\right\}_{d \in D}, \tag{5}$$

[1] $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ has the K-limit property if its graph contains all of its Komlos limits (i.e., if its graph is K-closed). $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ takes decomposable values in $\mathcal{L}^{\infty}_{Y}$ if for each $v \in \mathcal{L}^{\infty}_{Y}$, $U^0_{(\cdot)}$ and $U^1_{(\cdot)}$ in $\mathcal{S}^{\infty}(\mathcal{P}_v)$,

$$U^0_{(\cdot)} I_E(\cdot) + U^1_{(\cdot)} I_{\Omega \backslash E}(\cdot) \in \mathcal{S}^{\infty}(\mathcal{P}_v),$$

where $I_E(\cdot)$ and $I_{\Omega \backslash E}(\cdot)$ are indicator functions and $E \in B_{\Omega}$.

[2] If a set of one-shot Nash payoffs is homeomorphic to the unit circle it is said to be a set of circular one-shot Nash payoffs.

where $\omega \longrightarrow \Phi_d(\omega)$ is player $d's$ measurable constraint correspondence and $(x_d, x_{-d}) \longrightarrow U_d(\omega, v_d, x_d, x_{-d})$, is player $d's$ payoff function, given by

$$\begin{aligned} &U_d(\omega, v_d, x_d, x_{-d}) \\ &= (1-\beta_d) r_d(\omega, x_d, x_{-d}) + \beta_d \int_\Omega v_d(\omega') h(\omega'|\omega, x_d, x_{-d}) d\mu(\omega'), \end{aligned} \tag{6}$$

where $x := (x_d, x_{-d})$ is the profile of players' actions, $v_d \in \mathcal{L}^\infty_{Y_d}$ is player $d's$ value function summarizing player $d's$ state-contingent valuations (or prices), (ω, ω') are the current and coming states respectively, β_d is player $d's$ discount rate, $(x_d, x_{-d}) \longrightarrow r_d(\omega, x_d, x_{-d})$ is player $d's$ immediate payoff function, and $h(d\omega'|\omega, x)$ is the game's probability density (with respect to μ) over the coming states, ω', given current state ω and action profile x. Here the underlying state space is given by (Ω, B_Ω, μ), where Ω is a complete separable metric space equipped with the Borel σ-field and a nonatomic probability measure μ.

Under assumptions $[OSG]$ listed below, in a (ω, v)-game, player $d's$ feasible set of actions in state ω, $\Phi_d(\omega)$, is compact and **convex**, and player $d's$ payoff function, $U_d(\omega, \cdot, \cdot)$, in state ω is jointly continuous in (v_d, x) for each d and ω. Moreover, for each d and ω, $U_d(\omega, \cdot, x)$ is affine in $v_d \in \mathcal{L}^\infty_{Y_d}$ and $U_d(\omega, v_d, \cdot, x_{-d})$ is affine in $x_d \in X_d$. Thus, each player's payoff function, is a uniformly bounded, affinely parameterized, Caratheodory function. jointly continuous in action profiles, $x = (x_1, \ldots, x_m)$ and affine in each player's own action.[3]

2.1 Assumptions

The convex $OSGs$ we will consider here consist of the following primitives and satisfy the following list of assumptions (the *usual assumptions*), labeled $[OSG]$(1)-(11):

(1) $D =$ the set of players, consisting of m players indexed by $d = 1, 2, \ldots, m$ and each having discount rate given by $\beta_d \in (0, 1)$.
(2) (Ω, B_Ω, μ), the state space where Ω is a complete separable metric spaces with metric ρ_Ω, equipped with the Borel σ-field, B_Ω, upon which is defined a ***nonatomic*** *dominating probability measure, μ.*[4]
(3) $Y := Y_1 \times \cdots \times Y_m$, is the space of players' potential payoff profiles, $U := (U_1, \ldots, U_m)$, such that for each player d, $Y_d := [-M, M]$, $M > 0$, and is

[3] Here, $\rho_{w^* \times X}$ denotes the sum metric,

$$\rho_{w^*} + \rho_X.$$

[4] A set $E \subset \Omega$ is an atom of Ω relative to μ if the following implication holds: if $\mu(E) > 0$, then $H \subset E$ implies that $\mu(H) = 0$ or $\mu(E - H) = 0$. If Ω contains no atoms relative to μ, then μ is said to be nonatomic. Because Ω, is a complete, separable metric space $\mu(\cdot)$ is nonatomic if and only if $\mu(\{\omega\}) = 0$ for all $\omega \in \Omega$ (see Hildenbrand, 1974, pp 44-45). Also, note that the σ-field, B_Ω is countably generated. All the results we present here remain valid if instead we assume that Ω is an abstract set, but one equipped with a countably generated σ-field, as in Nowak and Raghavan (1992) - see assumption (i) p. 521 (also see Ash, 1972).

equipped with the absolute value metric, $\rho_{Y_d}(U_d, U'_d) := |U_d - U'_d|$ *and* Y *is equipped with the sum metric,* $\rho_Y := \sum_d \rho_{Y_d}$.

(4) $X := X_1 \times \cdots \times X_m := \prod_d X_d \subset E := \prod_d E_d$, *is the set of player action profiles,* $x := (x_1, \ldots, x_m)$, *such that for each player* d, X_d *is a* ***convex****, compact metrizable subset of a locally convex Hausdorff topological vector space* E_d *and is equipped with a metric,* ρ_{X_d}, *compatible with the locally convex topology inherited from* E_d, *and* X *is equipped with the sum metric,* $\rho_X := \sum_d \rho_{X_d}$.[5]

(5) $\omega \longrightarrow \Phi_d(\omega)$, *is player* d'*s measurable action constraint correspondence, defined on* Ω *taking nonempty,* ***convex****,* ρ_{X_d}*-closed (and hence compact) values in* X_d.[6]

(6) $\omega \longrightarrow \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega)$, *players' measurable action profile constraint correspondence, defined on* Ω *taking nonempty, convex, and* ρ_X*-closed (hence compact) values in* X.

(7) $\mathcal{L}^{\infty}_{Y_d}$, *the collection of all* μ*-equivalence classes of measurable, essentially bounded (value) functions,* $v_d(\cdot)$, *defined on* Ω *with values in* Y_d *a.e.* $[\mu]$, *equipped with metric* $\rho_{w^*_d}$ *compatible with the weak star topology inherited from* $\mathcal{L}^{\infty}_R$.

(8) $\mathcal{L}^{\infty}_Y := \mathcal{L}^{\infty}_{Y_1} \times \cdots \times \mathcal{L}^{\infty}_{Y_m} \subset \mathcal{L}^{\infty}_{R^m}$, *the collection of all* μ*-equivalence classes of measurable (value) function profiles,* $v(\cdot) := (v_1(\cdot), \ldots, v_m(\cdot))$, *defined on* Ω *with values in* Y *a.e.* $[\mu]$, *equipped with the sum metric* $\rho_{w^*} := \sum_d \rho_{w^*_d}$ *compatible with the weak star product topology inherited from* $\mathcal{L}^{\infty}_{R^m}$. $\mathcal{L}^{\infty}_Y$ ***is the set of parameter profiles,*** $v = (v_1, \ldots, v_m)$.

(9) $\mathcal{S}^{\infty}(\Phi_d(\cdot))$, *the set of all* μ*-equivalence classes of measurable functions,* $x_d(\cdot) \in \mathcal{L}^{\infty}_{X_d}$, *defined on* Ω *such that in* $x_d(\omega) \in \Phi_d(\omega)$ *a.e.* $[\mu]$, *and*

$$\mathcal{S}^{\infty}(\Phi(\cdot)) = \mathcal{S}^{\infty}(\Phi_1(\cdot)) \times \cdots \times \mathcal{S}^{\infty}(\Phi_m(\cdot)) \tag{7}$$

the set of all μ*-equivalence classes of measurable profiles,* $x(\cdot) = (x_1(\cdot), \ldots, x_m(\cdot)) \in \mathcal{L}^{\infty}_X$, *defined on* Ω *such that*

$$x(\omega) \in \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega) \text{ a.e. } [\mu]. \tag{8}$$

(10) $r_d(\cdot,\cdot) : \Omega \times X \longrightarrow Y_d$ *is player* d'*s affine Caratheodory stage payoff function (i.e., for each* ω, $r_d(\omega, \cdot)$ *is* ρ_X*-continuous on* X, *for each* x, $r_d(\cdot, x)$ *is* (B_Ω, B_{Y_d})*-measurable on* Ω, *and*

$$x_d \longrightarrow r_d(\omega, (x_d, x_{-d})), \tag{9}$$

is affine in x_d *for each* (ω, x_{-d}).

[5] In Nowak and Raghavan (1992), $X_d = \Delta(A_d)$, where $\Delta(A_d)$ denotes the collection of all probability measures with support contained in player $d's$ compact metric action set, A_d. $\Delta(A_d)$ is a closed, convex, compact metrizable subset of the locally convex Hausdorff topological vector space $ca(A_d)$ of countably additive, finite signed Borel measures on A_d.

[6] In Nowak and Raghavan (1992), the state-contingent, action constraint correspondence, $\Phi_d(\cdot)$, is given by $\Delta(\Phi_d(\cdot))$. Because $\Phi_d(\cdot)$ is lower measurable (and hence measurable - due to compactness), we know by Theorem 3(ii) in Himmelberg and Van Vleck (1975) that $\Delta(\Phi_d(\cdot))$ is also lower measurable (and hence measurable).

(11) $q(\cdot|\cdot,\cdot):\Omega\times X\longrightarrow\Delta(\Omega)$ is the law of motion defined on $\Omega\times X$ taking values in the space of probability measures on Ω, having the following properties: (i) each probability measure, $q(\cdot|\omega,x)$, in the collection

$$Q(\Omega\times X):=\{q(\cdot|\omega,x):(\omega,x)\in\Omega\times X\} \tag{10}$$

is absolutely continuous with respect to μ the dominating probability measure, denoted $Q(\Omega\times X)<<\mu$, (ii) for each $E\in B_\Omega$, $q(E|\cdot,\cdot)$ is jointly measurable on $\Omega\times X$, and (iii) the collection of probability density functions,

$$H_\mu:=\{h(\cdot|\omega,x):(\omega,x)\in\Omega\times X\}, \tag{11}$$

of $q(\cdot|\omega,x)$ with respect to μ is such that for each current state ω and a.e. $[\mu]$ in coming states ω' the real-valued function

$$x:=(x_d,x_{-d})\longrightarrow h(\omega'|\omega,x_d,x_{-d}) \tag{12}$$

is continuous in x and affine in x_d a.e. $[\mu]$ in ω'.[7]

2.2 Comments

(1) Under the stochastic continuity assumptions made above, $[OSG](11)$, we have by Scheffee's Theorem (see Billingsley, 1986, Theorem 16.11) that for each $\omega\in\Omega$,

$$\left.\begin{array}{l}\left\|q(\cdot|\omega,x^n)-q(\cdot|\omega,x^0)\right\|_\infty\\ :=\sup_{E\in\mathcal{B}(\Omega)}|q(E|\omega,x^n)-q(E|\omega,x^*)|\\ \leq\int_\Omega|h(\omega'|\omega,x^n)-h(\omega'|\omega,x^*)|\,d\mu(\omega')\longrightarrow 0,\end{array}\right\} \tag{13}$$

for any sequence of action profiles $\{x^n\}_n$ in $\Phi(\omega)$ converging to $x^*\in\Phi(\omega)$ (i.e., for each $\omega\in\Omega$ the conditional density mapping, $x\longrightarrow h(\cdot|\omega,x)$, is continuous in L_1 norm with respect to action profiles x). Thus, by Scheffee's Theorem, the L_1 norm continuity of $x\longrightarrow h(\cdot|\omega,x)$ in each state ω is equivalent to the continuity of $x\longrightarrow q(E|\omega,x)$ in each state ω with respect to action profiles x uniformly in $E\in B_\Omega$.[8]

(2) Under assumptions $[OSG](11)$, if $v_d^n\underset{\rho_{w_d^*}}{\longrightarrow}v_d^*$ and $x^n\underset{\rho_X}{\longrightarrow}x^*$, then

$$\int_\Omega v_d^n(\omega')h(\omega'|\omega,x^n)d\mu(\omega')\underset{R}{\longrightarrow}\int_\Omega v_d^*(\omega')h(\omega'|\omega,x^*)d\mu(\omega'). \tag{14}$$

[7] The strong stochastic assumptions stated here are the same as those in Nowak and Raghavan (1992) - see assumption (v) and Remark 1, p. 521.

[8] Again, note that our convex OSG model includes the Nowak-Raghavan (1992) OSG model over behavioral strategies.

This is because

$$\left|\int_\Omega v_d^n(\omega')h(\omega'|\omega,x^n)d\mu(\omega') - \int_\Omega v_d^*(\omega')h(\omega'|\omega,x^*)d\mu(\omega')\right|$$

$$\leq \underbrace{\left|\int_\Omega v_d^n(\omega')h(\omega'|\omega,x^n)d\mu(\omega') - \int_\Omega v_d^n(\omega')h(\omega'|\omega,x^*)d\mu(\omega')\right|}_{(a)}$$

$$+\underbrace{\left|\int_\Omega v_d^n(\omega')h(\omega'|\omega,x^*)d\mu(\omega') - \int_\Omega v_d^*(\omega')h(\omega'|\omega,x^*)d\mu(\omega')\right|}_{(b)},$$

and under assumptions $[OSG](8)$ and (11),

$$(a) \leq M \left\| q(\cdot|\omega,x^n) - q(\cdot|\omega,x^*) \right\|_\infty \longrightarrow 0,$$

and because $v_d^n \underset{\rho_{w_d^*}}{\longrightarrow} v_d^*$ and $h(\cdot|\omega,x^*) \in \mathcal{L}_R^1$, where $\mathcal{L}_R^1$ is the norm dual of $\mathcal{L}_R^\infty$,

$$(b) = \left|\int_\Omega (v_d^n(\omega') - v_d^*(\omega'))h(\omega'|\omega,x^*)d\mu(\omega')\right| \longrightarrow 0.$$

2.3 Nash Equilibria and the Three Nash Correspondences

In a (ω,v)-game each player $d = 1,2,\ldots,m$, seeks to choose a feasible action, $x_d \in \Phi_d(\omega)$ so as to maximize $d's$ payoff - i.e., so as to solve the problem

$$\max_{x_d \in \Phi_d(\omega)} U_d(\omega, v_d, (x_d, x_{-d})),$$

given the state $\omega \in \Omega$, player $d's$ value function, $v_d \in \mathcal{L}_Y^\infty$, and the feasible actions, x_{-d}, chosen by other players.

A profile of player actions, $x^* = (x_1^*,\ldots,x_m^*) \in \Phi_1(\omega) \times \cdots \times \Phi_m(\omega) := \Phi(\omega)$, is a *Nash equilibrium* for the (ω,v)-game, $\mathcal{G}_{(\omega,v)}$, if for each player $d \in D$

$$U_d(\omega, v_d, (x_d^*, x_{-d}^*)) = \max_{x_d \in \Phi_d(\omega)} U_d(\omega, v_d, (x_d, x_{-d}^*)).$$

Under assumptions $[OSG]$ for each $(\omega,v) \in \Omega \times \mathcal{L}_Y^\infty$ the (ω,v)-game, $\mathcal{G}_{(\omega,v)}$, has a nonempty, ρ_X-compact set of Nash equilibria, $\mathcal{N}(\omega,v)$, and it is straightforward to show that the *Nash correspondence*,

$$\mathcal{N}(\cdot,\cdot) : \Omega \times \mathcal{L}_Y^\infty \longrightarrow P_f(X) \tag{15}$$

is upper Caratheodory, $(B_\Omega \times B_{w^*}, B_X)$-measurable in (ω,v) and ρ_{w^*}-ρ_X-upper semicontinuous in v with nonempty, ρ_X-compact values.

Given the one-shot game, i.e., the OSG, specified above, with upper Caratheodory (uC) Nash correspondence, $(\omega,v) \longrightarrow \mathcal{N}(\omega,v)$, the OSG's *uC Nash payoff correspondence*, $(\omega,v) \longrightarrow \mathcal{P}(\omega,v)$, is given by

$$(\omega,v) \longrightarrow \mathcal{P}(\omega,v) := U(\omega,v,\mathcal{N}(\omega,v)), \tag{16}$$

where

$$U(\omega, v, \mathcal{N}(\omega, v)) := \cup_{x \in \mathcal{N}(\omega,v)} \{(U_1(\omega, v_1, x), \ldots, U_m(\omega, v_m, x))\}.$$

The Nash payoff correspondence induces a *Nash payoff selection correspondence*,

$$v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v) := \mathcal{S}^{\infty}(\mathcal{P}(\cdot, v)), \tag{17}$$

where $\mathcal{S}^{\infty}(\mathcal{P}_v)$ is the set of all μ-equivalence classes of measurable (value) functions, $U_{(\cdot)}$, such that $U_\omega \in \mathcal{P}(\omega, v)$ a.e. $[\mu]$.

Our main objective is to show that, under assumptions $[OSG]$, the Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, has fixed points - i.e., therefore we propose to show that there exists $v^* \in \mathcal{L}^{\infty}_Y$ such that $v^* \in \mathcal{S}^{\infty}(\mathcal{P}_{v^*})$.

Summary of Our Approach and Results:

We will show that any one-shot game (OSG) satisfying assumptions $[OSG]$ naturally has a Nash payoff selection correspondences, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, having the K-limit property - and therefore is a K-correspondences. We say that the Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, has the K-limit property if for any sequence $\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ K-converging to $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y$, where $U^n_\omega \in \mathcal{P}(\omega, v^n)$ a.e. $[\mu]$ for each n, the K-limit, $\widehat{U}_{(\cdot)}$, is such that $\widehat{U}_\omega \in Ls\{U^n_\omega\}$ a.e. $[\mu]$. Here, $Ls\{U^n_\omega\}$ denotes the set of cluster points of the sequence, $\{U^n_\omega\}_n$, in $Y \subset R^m$. Because $\mathcal{P}(\omega, \cdot)$ is upper semicontinuous on $\mathcal{L}^{\infty}_Y$ for ω a.e. $[\mu]$, we have for any sequence, $\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ K-converging to $(\widehat{v}, \widehat{U}_{(\cdot)})$ that $Ls\{U^n_\omega\} \subset \mathcal{P}(\omega, \widehat{v})$ a.e. $[\mu]$. Thus, if $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ has the K-limit property, we have for any sequence, $\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ K-converging to $(\widehat{v}, \widehat{U}_{(\cdot)})$ that

$$\left.\begin{array}{c} \widehat{U}_\omega \in Ls\{U^n_\omega\} \subset \mathcal{P}(\omega, \widehat{v}) \text{ a.e. } [\omega]. \\ \text{implying that} \\ \widehat{U}_{(\cdot)} \in \mathcal{S}^{\infty}(Ls\{U^n_{(\cdot)}\}) \subset \mathcal{S}^{\infty}(\mathcal{P}_{\widehat{v}}), \end{array}\right\} \tag{18}$$

where $\mathcal{S}^{\infty}(Ls\{U^n_{(\cdot)}\})$ denotes the collection of μ-equivalence classes of a.e. selections of the Ls correspondence, $\mathcal{S}^{\infty}(Ls\{U^n_{(\cdot)}\})$, and $\mathcal{S}^{\infty}(\mathcal{P}_{\widehat{v}})$ denotes the collection of μ-equivalence classes of a.e. selections of the measurable part of the uC Nash payoff correspondence at value function profile $\widehat{v}$, $\omega \longrightarrow \mathcal{P}(\omega, \widehat{v})$. We then show that for any one-shot game where players have convex, compact metric action sets and player's payoff functions are given by

$$U_d(\omega, v_d, x_d, x_{-d})$$

$$:= (1 - \beta_d) r_d(\omega, x_d, x_{-d}) + \beta_d \int_\Omega v_d(\omega') h(\omega' | \omega, x_d, x_{-d}) d\mu(\omega')$$

satisfying assumptions $[OSG]$ above, then (18) holds. Given the near equivalence of K-convergence and weak star convergence (see 22 below), it is then a simple matter to show that the Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, is upper semicontinuous and given the decomposability of $\mathcal{S}^{\infty}(\mathcal{P}_v)$ for each $v \in \mathcal{L}^{\infty}_Y$ and the Lyapunov machinery made available by the nonatomicity of μ, it then becomes possible to show that $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ takes *contractible* values (with respect to the weak star topologies). Together, the upper semicontinuity and contractible valuedness of $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, imply that $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is approximable, and therefore, has fixed points.

Before we present our results, we review the notions of weak star convergence, Komlos convergence, and decomposability.

3 w^*-Convergence and K-Convergence in $\mathcal{L}^{\infty}_Y$

A sequence, $\{v^n\}_n \subset \mathcal{L}^{\infty}_Y$, converges weak star to $v^* = (v_1^*(\cdot), \ldots, v_m^*(\cdot)) \in \mathcal{L}^{\infty}_Y$, denoted by $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$, if and only if

$$\int_\Omega \langle v^n(\omega), l(\omega)\rangle_{R^m}\, d\mu(\omega) \longrightarrow \int_\Omega \langle v^*(\omega), l(\omega)\rangle_{R^m}\, d\mu(\omega) \tag{19}$$

for all $l(\cdot) \in \mathcal{L}^1_{R^m}$.

A sequence, $\{v^n\}_n \subset \mathcal{L}^{\infty}_Y$, K-convergences (i.e., Komlos convergence - Komlos, 1967) to $\widehat{v} \in \mathcal{L}^{\infty}_Y$, denoted by $v^n \underset{K}{\longrightarrow} \widehat{v}$, if and only if every subsequence, $\{v^{n_k}(\cdot)\}_k$, of $\{v^n(\cdot)\}_n$ has a sequence of arithmetic mean functions, $\{\widehat{v}^{n_k}(\cdot)\}_k$, where

$$\widehat{v}^{n_k}(\cdot) := \frac{1}{k}\sum_{q=1}^{k} v^{n_q}(\cdot), \tag{20}$$

such that

$$\widehat{v}^{n_k}(\omega) \underset{R^m}{\longrightarrow} v^*(\omega) \ \ a.e.\ [\mu]. \tag{21}$$

The relationship between w^*-convergence and K-convergence is summarized via the following results (see Theorem A 2.1, Page, 2016): For every sequence of value functions, $\{v^n\}_n \subset \mathcal{L}^{\infty}_Y$, and $\widehat{v}$ and v^* in $\mathcal{L}^{\infty}_Y$ the following statements are true:

$$\left.\begin{array}{c}\text{(i) If the sequence } \{v^n\}_n\ K\text{-converges to } \widehat{v} \in \mathcal{L}^{\infty}_Y, \\ \text{then } \{v^n\}_n\ \rho_{w^*}\text{-converges to } v^* \in \mathcal{L}^{\infty}_Y \text{ and } \widehat{v}(\omega) = v^*(\omega) \text{ a.e. } [\mu]. \\ \\ \text{(ii) If the sequence } \{v^n\}_n\ \rho_{w^*}\text{-converges to } v^* \in \mathcal{L}^{\infty}_Y, \text{ then} \\ \text{every subsequence } \{v^{n_k}\}_k \text{ of } \{v^n\}_n \\ \text{has a further subsequence, } \{v^{n_{k_r}}\}_r,\ K\text{-converging to to } \widehat{v} \in \mathcal{L}^{\infty}_Y, \\ \text{and } v^*(\omega) = \widehat{v}(\omega) \text{ a.e. } [\mu].\end{array}\right\} \tag{22}$$

For any sequence of value function profiles, $\{v^n\}_n$, in $\mathcal{L}^{\infty}_Y$ it is automatic that

$$sup_n \int_\Omega \|v^n(\omega)\|_{R^m}\, d\mu(\omega) < +\infty. \tag{23}$$

Thus, by the classical Komlos Theorem (1967), any such sequence, $\{v^n\}_n$, has a subsequence, $\{v^{n_k}\}_k$ that K-converges to some K-limit, $v^* \in \mathcal{L}_Y^\infty$. By Page (1991) Proposition 1(1)

$$\widehat{v}(\omega) \in coLs\{v^n(\omega)\} \ a.e. \ [\mu]$$

and Proposition 1(2) there exists an integrable R^m-valued function, $v^*(\cdot)$, such that $v^*(\omega) \in Ls\{v^n(\omega)\}$ a.e. $[\mu]$ and

$$\int_\Omega v^*(\omega)d\mu(\omega) = \int_\Omega \widehat{v}(\omega)d\mu(\omega).$$

Moreover, by Proposition 2 in Page (1991). if $lim_n \int_\Omega v^n(\omega)d\mu(\omega)$ exists, then the usual Fatou's Lemma in Several Dimensions holds, and we have

$$lim_n \int_\Omega v^n(\omega)d\mu(\omega) = \int_\Omega v^*(\omega)d\mu(\omega) = \int_\Omega \widehat{v}(\omega)d\mu(\omega).$$

4 Decomposability in $\mathcal{L}_Y^\infty$

A subset $\mathcal{S}$ of $\mathcal{L}_Y^\infty$ is said to be decomposable if for any two functions $U_{(\cdot)}^0$ and $U_{(\cdot)}^1$ in $\mathcal{S}$ and for any $E \in B_\Omega$, we have

$$U_{(\cdot)}^0 I_E(\cdot) + U_{(\cdot)}^1 I_{\Omega \backslash E}(\cdot) \in \mathcal{S}.$$

For any uC Nash payoff correspondence, $\mathcal{P}(\cdot,\cdot) : \Omega \times \mathcal{L}_Y^\infty \longrightarrow P_f(Y)$, the induced Nash payoff selection correspondence, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, takes decomposable values. Moreover, for each v, $\mathcal{S}^\infty(\mathcal{P}_v)$ is $\|\cdot\|_1$-closed (or $\mathcal{L}_{R^m}^1$-closed) in $\mathcal{L}_{R^m}^\infty$. Thus, for any sequence $\{U_{(\cdot)}^n\}_n$ in $\mathcal{S}^\infty(\mathcal{P}_v)$ converging in $\mathcal{L}_{R^m}^1$-norm to $U_{(\cdot)}^0 \in \mathcal{L}_{R^m}^\infty$, we have $U_{(\cdot)}^0 \in \mathcal{S}^\infty(\mathcal{P}_v)$. We will denote by $cl_1\mathcal{S}^\infty(\mathcal{P}_v)$ the $\mathcal{L}_{R^m}^1$-closure of $\mathcal{S}^\infty(\mathcal{P}_v)$ in $\mathcal{L}_{R^m}^\infty$. By Lemma 1 in Pales and Zeidan (1999), we know that, in addition to $\mathcal{S}^\infty(\mathcal{P}_v)$ being decomposable, $\mathcal{S}^\infty(\mathcal{P}_v)$ is $\mathcal{L}_{R^m}^1$-closed in $\mathcal{L}_{R^m}^\infty$. Thus, we have

$$cl_1\mathcal{S}^\infty(\mathcal{P}_v) = \mathcal{S}^\infty(\mathcal{P}_v).$$

We also know by Corollary 1 in Pales and Zeidan (1999) that

$$cl_1\mathcal{S}^\infty(\mathcal{P}_v)$$

$$= \left\{ U_{(\cdot)} \in \mathcal{L}_{R^m}^\infty : \exists \ \{U_{(\cdot)}^n\}_n \ \subset \mathcal{S}^\infty(\mathcal{P}_v) \text{ such that } \lim_n \left\| U_{(\cdot)}^n - U_{(\cdot)} \right\|_1 = 0 \right\}.$$

Finally, note that $\mathcal{L}_Y^\infty$ is $\mathcal{L}_{R^m}^1$-closed in $\mathcal{L}_{R^m}^\infty$ and decomposable.

5 The K-Limit Property

Let $(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := U(\omega, v, \mathcal{N}(\omega, v))$ be a Nash payoff correspondence induced by a Nash correspondence belonging to a convex OSG satisfying assumptions $[OSG]$ and let

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v)$$

be the induced Nash payoff selection correspondence. We have the following formal definition of the K-limit property.

Definition 1 *(The K-Limit Property and K-Correspondences):*
We say that the Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, has the K-limit property if for any K-converging sequence,

$$\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}),$$

the K limit, $(\widehat{v}, \widehat{U}_{(\cdot)})$, is such that

$$\widehat{U}_{\omega} \in Ls\{U^n_{\omega}\} \subset \mathcal{P}(\omega, \widehat{v}) \text{ a.e. } [\mu].$$

Because $\{(v^n, U^n_{(\cdot)})\}_n$ K-converges to $(\widehat{v}, \widehat{U}_{(\cdot)})$, $\{v^n\}_n$ ρ_{w^*}-converges to $\widehat{v}$ and $\{U^n_{(\cdot)})_n$ ρ_{w^*}-converges to $\widehat{U}_{(\cdot)}$. Moreover, because $\mathcal{P}(\omega, \cdot)$ is ρ_{w^*}-ρ_Y-upper semicontinuous, $(v^n, U^n_{\omega}) \in Gr\mathcal{P}(\omega, \cdot)$ *a.e.* $[\mu]$ implies that $Ls\{U^n_{\omega}\} \subset \mathcal{P}(\omega, \widehat{v})$ *a.e.* $[\mu]$. Thus, if the Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, has the K-limit property, and therefore is a K-correspondence, then for *any K-converging sequence,*

$$\{(v^n, U^n_{(\cdot)})\}_n \subset \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y,$$

with $(v^n, U^n_{(\cdot)}) \in Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ for each n, the K limit, $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y$, is such that

$$\widehat{U}_{(\cdot)} \in \mathcal{S}^{\infty}(Ls\{U^n_{(\cdot)}\}) \subset \mathcal{S}^{\infty}(\mathcal{P}_{\widehat{v}}).$$

Let N^{∞} be the exceptional set (i.e., the set of μ-measure zero) such for $\omega \in \Omega \backslash N^{\infty}$, $U^n_{\omega} \in \mathcal{P}(\omega, v^n)$ for all n. For each n, we have by the Measurable Implicit Function Theorem (e.g., Himmelberg, 1975, Theorem 7.1) a (B_{Ω}, B_X)-measurable function, $x^n(\cdot) : \Omega \longrightarrow X$, such that for each n and $\omega \in \Omega \backslash N^{\infty}$,

$$U^n_{\omega} = U(\omega, v^n, x^n(\omega)) \in \mathcal{P}(\omega, v^n) \text{ with } x^n(\omega) \in \mathcal{N}(\omega, v^n),$$

and thus,

$$\{(v^n, U(\cdot, v^n, x^n(\cdot))\}_n \subset Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) \subset \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y.$$

An alternative statement of the K-limit property is

$\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ has the K-limit property, and therefore, is a K-correspondence, if for any K-converging sequence,

$$\{(v^n, U(\cdot, v^n, x^n(\cdot)))\}_n \subset Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) \subset \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y,$$

with K-limit, $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y$, the K-limit, $(\widehat{v}, \widehat{U}_{(\cdot)})$, is such that

$$\widehat{U}_{\omega} \in U(\omega, \widehat{v}, Ls\{x^n(\omega)\}) \subset \mathcal{P}(\omega, \widehat{v}) \text{ a.e. } [\mu],$$

where

$$U(\omega, \widehat{v}, Ls\{x^n(\omega)\}) := \{U(\omega, \widehat{v}, x) \in Y : x \in Ls\{x^n(\omega)\}\}.$$

We note that $Ls\{U^n_{\omega}\} = U(\omega, \widehat{v}, Ls\{x^n(\omega)\})$.

Now we have our main result on the K-limit property for nonatomic convex $DSGs$.

Theorem 1 (*The K-Limit Theorem for nonatomic convex DSGs - the Nice Lemma*)
Let

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := U(\omega, v, \mathcal{N}(\omega, v))$$

$$= \left\{ \left((1-\beta_d) r_d(\omega, x) + \beta_d \int_\Omega v_d(\omega') h(d\omega'|\omega, x) d\mu(\omega') \right)_d : x \in \mathcal{N}(\omega, v) \right\},$$

be the Nash payoff sub-correspondence induced a Nash correspondence belonging to a nonatomic convex DSG. Then the induced Nash payoff selection correspondence, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, has the K-limit property.

Proof: Let $\{(v^n, U^n_{(\cdot)})\}_n \subset \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$ be any K-converging sequence with K-limit, $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$, where for each n and ω a.e. $[\mu]$,

$$\left. \begin{array}{c} U^n_\omega = U(\omega, v^n, x^n(\omega)) \in \mathcal{P}(\omega, v^n) \\ \text{and} \\ x^n(\omega) \in \mathcal{N}(\omega, v^n), \end{array} \right\} \tag{24}$$

with

$$U^n_\omega = U(\omega, v^n, x^n(\omega))$$
$$= \left((1-\beta_d) r_d(\omega, x^n(\omega)) + \beta_d \int_\Omega v^n_d(\omega') h(\omega'|\omega, x^n(\omega)) d\mu(\omega') \right)_{d \in D}. \tag{25}$$

Let $\{v^n\}_n$ and $\{U^n_{(\cdot)}\}_n$ be sequences in $\mathcal{L}^\infty_Y$, and let

$$\widehat{v}^n(\cdot) := \frac{1}{n} \sum_{k=1}^{n} v^k(\cdot) \text{ and } \widehat{U}^n_{(\cdot)} := \frac{1}{n} \sum_{k=1}^{n} U^k_{(\cdot)} \tag{26}$$

denote the arithmetic mean functions induced by the sequences, $\{v^n\}_n$ and $\{U^n_{(\cdot)}\}_n$. Because $\{v^n\}_n$ and $\{U^n_{(\cdot)}\}_n$ K-converge to $\widehat{v}$ and $\widehat{U}_{(\cdot)}$ in $\mathcal{L}^\infty_Y$, any sequence of arithmetic mean functions, $\{\widehat{v}^{n_k}\}_k$ and $\{\widehat{U}^{n_k}_{(\cdot)}\}_k$ belonging to any subsequences, $\{v^{n_k}\}_k$ and $\{U^{n_k}_{(\cdot)}\}_k$ of $\{v^n\}_n$ and $\{U^n_{(\cdot)}\}_n$, respectively, converge pointwise a.e. to $\widehat{v}$ and $\widehat{U}_{(\cdot)}$, respectively, implying that the sequences themselves, $\{v^n\}_n$ and $\{U^n_{(\cdot)}\}_n$, ρ_{w^*}-converge to $\widehat{v}$ and $\widehat{U}_{(\cdot)}$, respectively. We note that the set of μ-measure zero off of which pointwise arithmetic mean convergences takes place depends on the subsequences over which arithmetic means are computed. With this being noted, consider the following: For each n let

$$G^n(\omega, \omega') := \left((1-\beta_d) r_d(\omega, x^n(\omega)) + \beta_d v^n_d(\omega') h(\omega'|\omega, x^n(\omega)) \right)_d, \tag{27}$$

and note that

$$U_\omega^n := \int_\Omega G^n(\omega, \omega')d\mu(\omega'). \tag{28}$$

Because $|r_d(\omega, x^n(\omega))| \leq M$ and $|v_d^n(\omega')| \leq M$ and because $h(\cdot|\omega, x^n(\omega))$ is a probability density, we have for all d, n and (ω, ω') that the sequence of functions, $\{G^n(\cdot,\cdot)\}_n \subset \mathcal{L}^1_{R^m}(\Omega \times \Omega)$, is norm bounded. By Komlos (1967), there is a subsequence, $\{G^{n_k}(\cdot,\cdot)\}_k$, K-converging to $\widehat{G}(\cdot,\cdot) \in \mathcal{L}^1_{R^m}(\Omega \times \Omega)$. Consider the sequence of arithmetic mean functions, $\{\widehat{G}^{n_k}(\cdot,\cdot)\}_k$, induced by the subsequence, $\{G^{n_k}(\cdot,\cdot)\}_k$, where

$$\widehat{G}^{n_k}(\cdot,\cdot) := \frac{1}{k}\sum_{q=1}^{k} G^{n_q}(\cdot,\cdot).$$

We have that $\widehat{G}^{n_k}(\omega,\omega') \longrightarrow_{R^m} \widehat{G}(\omega,\omega')$ for (ω,ω') off an exceptional set $\widehat{E}^{n_k} \in B_\Omega \times B_\Omega$ with $\lambda(\widehat{E}^{n_k}) = 0$, where λ is the product probability measure given by $\lambda := \mu \otimes \mu$. For the exceptional set $\widehat{E}^{n_k}$ we have by the Product Measure Theorem (Ash 2.6.2, 1972) that

$$\lambda(\widehat{E}^{n_k}) = \int_\Omega \mu(\widehat{E}^{n_k}(\omega))d\mu(\omega) = 0,$$

where

$$\widehat{E}^{n_k}(\omega) := \left\{\omega' \in \Omega : (\omega,\omega') \in \widehat{E}^{n_k}\right\},$$

implying that for some $\widehat{N}_G^{n_k}$ with $\mu(\widehat{N}_G^{n_k}) = 0$, $\mu(\widehat{E}^{n_k}(\omega)) = 0$ for all $\omega \in \Omega \backslash \widehat{N}_G^{n_k}$. Thus for each $\omega \in \Omega \backslash \widehat{N}_G^{n_k}$

$$\widehat{G}^{n_k}(\omega,\omega') \longrightarrow \widehat{G}(\omega,\omega') \text{ for } \omega' \text{ a.e. } [\mu],$$

Also, let $\widehat{N}^{n_k}$ be the exceptional set off of which $\{\widehat{U}_{(\cdot)}^{n_k}\}_k$ converges pointwise to $\widehat{U}_{(\cdot)}$ (i.e., $\widehat{U}_\omega^{n_k} \longrightarrow \widehat{U}_\omega$ for all $\omega \in \Omega \backslash \widehat{N}^{n_k}$). Thus, we have for all $\omega \in \Omega \backslash (\widehat{N}^{n_k} \cup \widehat{N}_G^{n_k})$ that

$$\frac{1}{k}\textstyle\sum_{q=1}^{k} U_\omega^{n_q} = \frac{1}{k}\sum_{q=1}^{k}\int_\Omega G^{n_q}(\omega,\omega')d\mu(\omega')$$

$$= \textstyle\int_\Omega \frac{1}{k}\sum_{q=1}^{k} G^{n_q}(\omega,\omega')d\mu(\omega') = \int_\Omega \widehat{G}^{n_k}(\omega,\omega')d\mu(\omega')$$

$$\longrightarrow \textstyle\int_\Omega \widehat{G}(\omega,\omega')d\mu(\omega') = \widehat{U}_\omega.$$

Next let $U_{(\cdot)}^*$ be any measurable selection of the Ls correspondence, $\omega \longrightarrow Ls\{U_\omega^{n_k}\}$, (a selection whose existence is guaranteed by Kuratowski-Ryll-Nardzewski, 1965). By the Measurable Implicit Function Theorem (e.g., Himmelberg, 1975, Theorem 7.1) there exists an everywhere measurable selection, $x^*(\cdot)$ of the Ls correspondence, $Ls\{x^{n_k}(\cdot)\}$, such that

$$U_\omega^* = \left((1-\beta_d)r_d(\omega, x^*(\omega)) + \beta_d \textstyle\int_\Omega \widehat{v}_d(\omega')h(\omega'|\omega, x^*(\omega))d\mu(\omega')\right)_{d\in D}$$

$$\in Ls\{\textstyle\int_\Omega G^n(\omega,\omega')d\mu(\omega')\}.$$

Thus, for each $\omega \in \Omega$ there is some further subsequence, $\{x^{n_{k_r}}(\omega)\}_r$ ρ_X-converging to $x^*(\omega)$ and given that $v^n \underset{\rho_{w^*}}{\longrightarrow} \widehat{v}$, we have that

$$\left.\begin{aligned} U_\omega^{n_{k_r}} &= \left((1-\beta_d)r_d(\omega, x^{n_{k_r}}(\omega)) + \beta_d \int_\Omega v_d^{n_{k_r}}(\omega')h(\omega'|\omega, x^{n_{k_r}}(\omega))d\mu(\omega')\right)_d \\ &\longrightarrow \left((1-\beta_d)r_d(\omega, x^*(\omega)) + \beta_d \int_\Omega \widehat{v}_d(\omega')h(\omega'|\omega, x^*(\omega))d\mu(\omega')\right)_d \\ &= U_\omega^* \in Ls\{\int_\Omega G^n(\omega,\omega')d\mu(\omega') = Ls\{U_\omega^{n_k}\} \subset Ls\{U_\omega^n\}. \end{aligned}\right\} \tag{29}$$

Now consider the auxiliary function

$$F^n(\omega,\omega') := ((1-\beta_d)r_d(\omega, x^n(\omega)) + \beta_d\widehat{v}_d(\omega')h(\omega'|\omega, x^n(\omega)))_d\,, \tag{30}$$

where for each d the value function, $v_d^n(\cdot)$, in the definition of the function $G^n(\cdot,\cdot)$ (see expression (27)) is replaced by the K-limit (i.e., the weak star limit), $\widehat{v}_d(\cdot)$, in the definition of the function $F^n(\cdot,\cdot)$ in expression (30).[9] Note that for any $(\omega,\omega') \in \Omega\times\Omega$, and for any subsequence, $\{x^{n_k}(\cdot)\}_k$, if $x^{n_k}(\omega) \underset{\rho_X}{\longrightarrow} x^*(\omega)$, then

$$\left.\begin{aligned} &F^{n_k}(\omega,\omega') \longrightarrow F^*(\omega,\omega') \\ &:= ((1-\beta_d)r_d(\omega, x^*(\omega)) + \beta_d\widehat{v}_d(\omega')h(\omega'|\omega, x^*(\omega)))_d. \end{aligned}\right\} \tag{31}$$

Thus, given that $v^n \underset{\rho_{w^*}}{\longrightarrow} \widehat{v}$, we have that $F^*(\cdot,\cdot)$ is an everywhere measurable selection of the Ls correspondence, $Ls\{F^{n_k}(\cdot,\cdot)\}$ if and only if there is an everywhere measurable selection, $x^*(\cdot)$, of the Ls correspondence, $Ls\{x^{n_k}(\cdot)\}$, such that for all k and (ω,ω')

$$F^{n_k}(\omega,\omega') = \left((1-\beta_d)r_d(\omega, x^{n_k}(\omega)) + \beta_d\widehat{v}_d(\omega')h(\omega'|\omega, x^{n_k}(\omega))\right)_d\,,$$

and if at ω, $x^{n_{k_r}}(\omega) \underset{\rho_X}{\longrightarrow} x^*(\omega)$, for some further subsequence, then we have at this ω,

$$\left.\begin{aligned} &F^{n_{k_r}}(\omega,\omega') \longrightarrow ((1-\beta_d)r_d(\omega, x^*(\omega)) + \beta_d\widehat{v}_d(\omega')h(\omega'|\omega, x^*(\omega)))_d \\ &:= F^*(\omega,\omega') \text{ a.e. } [\mu] \text{ in } \omega'. \end{aligned}\right\} \tag{32}$$

and in general we have for each $\omega \in \Omega$, $\int_\Omega F^{n'_k}(\omega,\omega') \underset{\rho_Y}{\longrightarrow} \int_\Omega F^*(\omega,\omega')$ where $\{x^{n'_k}(\omega)\}_{n'_k}$ is any further subsequence of $\{x^{n_k}(\omega)\}_k$ such that $x^{n'_k}(\omega) \underset{\rho_X}{\longrightarrow} x^*(\omega)$.

We have the following observations:

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[9] Because $\{v^n\}_n$ K-converges to $\widehat{v}$, any sequence of arithmetic mean functions, $\{\widehat{v}^{n_k}\}_k$, induced by any subsequence, $\{v^{n_k}\}_k$, of $\{v^n\}_n$, converges pointwise a.e. $[\mu]$ to $\widehat{v}$ - i.e., $\widehat{v}^{n_k}(\omega) \longrightarrow \widehat{v}(\omega)$ a.e. $[\mu]$, and $\widehat{v}^{n_k} \underset{\rho_{w^*}}{\longrightarrow} \widehat{v}$. Moreover, the subsequence itself ρ_{w^*}-converges to $\widehat{v}$ - i.e., $v^{n_k} \underset{\rho_{w^*}}{\longrightarrow} \widehat{v}$.

We have already that for any everywhere measurable selection, $x^*(\cdot)$, of the correspondence, $\omega \longrightarrow Ls\{x^{n_k}(\omega)\}$, (whose existence is also guaranteed by Kuratowski-Ryll-Nardzewski, 1965), if at $\omega \in \Omega\backslash(\widehat{N}^{n_k} \cup \widehat{N}_G^{n_k})$, $x^{n'_k}(\omega) \underset{\rho_X}{\longrightarrow} x^*(\omega)$, for some further subsequence, $\{x^{n'_k}(\omega)\}_r$, then by (14) and (29)

$$\left.\begin{array}{c} \int_\Omega G^{n'_k}(\omega,\omega')d\mu(\omega') \\ = \Big((1-\beta_d)r_d(\omega, x^{n'_k}(\omega)) + \beta_d \int_\Omega v_d^{n'_k}(\omega')h(\omega'|\omega, x^{n'_k}(\omega))d\mu(\omega')\Big)_d, \\ \longrightarrow \big((1-\beta_d)r_d(\omega, x^*(\omega)) + \beta_d \int_\Omega \widehat{v}_d(\omega')h(\omega'|\omega, x^*(\omega))d\mu(\omega')\big)_d. \end{array}\right\} \tag{33}$$

Also, we have at $\omega \in \Omega\backslash(\widehat{N}^{n_k} \cup \widehat{N}_G^{n_k})$, where $x^{n'_k}(\omega) \underset{\rho_X}{\longrightarrow} x^*(\omega)$, that

$$\left.\begin{array}{c} F^{n'_k}(\omega,\omega') = \Big((1-\beta_d)r_d(\omega, x^{n'_k}(\omega)) + \beta_d\widehat{v}_d(\omega')h(\omega'|\omega, x^{n'_k}(\omega))\Big)_d \\ \longrightarrow ((1-\beta_d)r_d(\omega, x^*(\omega)) + \beta_d\widehat{v}_d(\omega')h(\omega'|\omega, x^*(\omega)))_d \\ := F^*(\omega,\omega') \text{ a.e. } [\mu] \text{ in } \omega', \end{array}\right\} \tag{34}$$

implying that,

$$\left.\begin{array}{c} \int_\Omega F^{n'_k}(\omega,\omega')d\mu(\omega') \\ = \Big((1-\beta_d)r_d(\omega, x^{n'_k}(\omega)) + \beta_d \int_\Omega \widehat{v}_d(\omega')h(\omega'|\omega, x^{n'_k}(\omega))d\mu(\omega')\Big)_d, \\ \overset{n'_k}{\longrightarrow} \big((1-\beta_d)r_d(\omega, x^*(\omega)) + \beta_d \int_\Omega \widehat{v}_d(\omega')h(\omega'|\omega, x^*(\omega))d\mu(\omega')\big)_d \\ = \int_\Omega F^*(\omega,\omega')d\mu(\omega'). \end{array}\right\} \tag{35}$$

Thus, we have that,

$$\lim_r \int_\Omega G^{n'_k}(\omega,\omega')d\mu(\omega') = \int_\Omega (\lim_r F^{n'_k}(\omega,\omega'))d\mu(\omega'), \tag{36}$$

and in general, we have for $\omega \in \Omega$ that

$$Ls\left\{\int_\Omega G^{n_k}(\omega,\omega')d\mu(\omega')\right\} = \int_\Omega Ls\{F^{n_k}(\omega,\omega')\}d\mu(\omega'). \tag{37}$$

Note that while the strategies, $x^n(\cdot)$, in the sequence, $\{x^n(\cdot)\}_n$, are, state by state for $\omega \in \Omega\backslash(\widehat{N}^{n_k} \cup \widehat{N}_G^{n_k})$ Nash equilibria relative to the sequence of value function profiles, $\{v^n(\cdot)\}_n$, in (27), they may not be state by state Nash equilibria relative to the valuation function profile, $\widehat{v}(\cdot)$, in (30) - *except in the limit* (i.e., $x^*(\cdot)$ is, state by state for $\omega \in \Omega\backslash(\widehat{N}^{n_k} \cup \widehat{N}_G^{n_k})$ a Nash equilibrium relative to the K-limit valuation function profile, $\widehat{v}(\cdot)$, appearing in both (33) and (35)).

Finally, by Page (1991) Proposition 1(1), we have for $\omega \in \Omega \backslash (\widehat{N}^{n_k} \cup \widehat{N}_G^{n_k})$ that

$$\widehat{U}_\omega \in coLs\{U_\omega^{n_k}\} = coLs\{\int_\Omega G^{n_k}(\omega, \omega')d\mu(\omega')\}, \tag{38}$$

and therefore by (37) we have for $\omega \in \Omega \backslash (\widehat{N}^{n_k} \cup \widehat{N}_G^{n_k})$ that

$$\left.\begin{array}{c} \widehat{U}_\omega \in coLs\{U_\omega^{n_k}\} = coLs\{\int_\Omega G^{n_k}(\omega, \omega')d\mu(\omega')\} \\ = co\int_\Omega Ls\{F^{n_k}(\omega, \omega')\}d\mu(\omega'). \end{array}\right\} \tag{39}$$

By the properties of Aumann integrals over nonatomic probability spaces (see Hildenbrand, 1974), we have that

$$co\int_\Omega Ls\{F^{n_k}(\omega, \omega')\}d\mu(\omega') = \int_\Omega Ls\{F^{n_k}(\omega, \omega')\}d\mu(\omega'). \tag{40}$$

and again by (37) we have that

$$\left.\begin{array}{c} \int_\Omega Ls\{F^{n_k}(\omega, \omega')\}d\mu(\omega') = Ls\{\int_\Omega G^{n_k}(\omega, \omega')d\mu(\omega')\} \\ = Ls\{U_\omega^{n_k}\} \subset Ls\{U_\omega^n\}. \end{array}\right\} \tag{41}$$

Thus, by Proposition 1(1) in Page (1991) and (37)-(41) above, we have for $\omega \in \Omega \backslash (\widehat{N}^{n_k} \cup \widehat{N}_G^{n_k})$ that

$$\left.\begin{array}{c} \widehat{U}_\omega \in coLs\{U_\omega^{n_k}\} = coLs\{\int_\Omega G^{n_k}(\omega, \omega')d\mu(\omega')\} \\ = \int_\Omega Ls\{F^{n_k}(\omega, \omega')\}d\mu(\omega') \\ = Ls\{\int_\Omega G^{n_k}(\omega, \omega')d\mu(\omega')\} = Ls\{U_\omega^{n_k}\} \subset Ls\{U_\omega^n\}. \end{array}\right\} \tag{42}$$

Finally, for each $\omega \in \Omega$, because $\mathcal{P}(\omega, \cdot)$ is upper semicontinuous, we have for $\omega \in \Omega \backslash (\widehat{N}^{n_k} \cup \widehat{N}_G^{n_k})$ that

$$\widehat{U}_\omega \in Ls\{U_\omega^n\} = U(\omega, \widehat{v}, Ls\{x^n(\omega)\}) \subset \mathcal{P}(\omega, \widehat{v}). \tag{43}$$

We can conclude, therefore, that in a nonatomic, convex DSG, if we are given any K-converging sequence $\{(v^n, U_{(\cdot)}^n)\}_n$ with K-limit $(\widehat{v}, \widehat{U}_{(\cdot)})$, where for each n

$$U_\omega^n = U(\omega, v^n, x^n(\omega)) \in \mathcal{P}(\omega, v^n) \text{ and } x^n(\omega) \in \mathcal{N}(\omega, v^n) \text{ a.e. } [\mu],$$

then there exists for each ω, off some exceptional set of measure zero, a $U_\omega^* \in Ls\{U_\omega^n\}$ such that $U_\omega^* = \widehat{U}_\omega$ - implying that

$$\left.\begin{array}{c} \widehat{U}_\omega \in \mathcal{P}(\omega, \widehat{v}) \\ \text{so that} \\ \widehat{U}_{(\cdot)} \in \mathcal{S}^\infty(\mathcal{P}_{\widehat{v}}). \end{array}\right\} \tag{44}$$

Thus, the Nash payoff selection sub-correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, belonging to any nonatomic convex DSG has the K-limit property, and therefore, is a K-correspondence.
Q.E.D.

By Theorem 1 above, the K-limit $\widehat{U}_{(\cdot)}$ of a K-converging sequence of Nash payoff selections, $\{U^n_{(\cdot)}\}_n$, of a nonatomic convex DSG is an a.e. measurable selection of the Ls correspondence, $Ls\{U^n_{(\cdot)}\}$, induced by the sequence.

6 A Fixed Point Theorem for Nonatomic Convex $DSGs$

Let $(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := U(\omega, v, \mathcal{N}(\omega, v))$ be a Nash payoff correspondence induced by a Nash correspondence belonging to a nonatomic convex DSG. We will show that because the uC Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, induced by a Nash correspondence belonging to a nonatomic convex DSG has the K-limit property, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ *is a* ρ_{w^*}-ρ_{w^*}*-USCO taking contractible values* - implying that $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is ρ_{w^*}-ρ_{w^*}-approximable and therefore that $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ has fixed points.[10]

6.1 The Contractibility Result

Theorem 2 *($\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a ρ_{w^*}-ρ_{w^*}-USCO taking contractible values in $\mathcal{L}^{\infty}_Y$)*
Let

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := U(\omega, v, \mathcal{N}(\omega, v))$$

be the Nash payoff correspondence induced by a Nash correspondence belonging to a nonatomic convex DSG. Then $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a ρ_{w^*}-ρ_{w^*}-USCO and for each $v \in \mathcal{L}^{\infty}_Y$, $\mathcal{S}^{\infty}(\mathcal{P}_v)$ is contractible.

Proof: First, by Theorem 1 above, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a K-correspondence, under assumptions $[OSG]$(1)-(11) it follows from Komlos (1967) and Page (1991) that for each $v \in \mathcal{L}^{\infty}_Y$, $\mathcal{S}^{\infty}(\mathcal{P}_v)$ is ρ_{w^*}-compact. Therefore, to show that $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a ρ_{w^*}-ρ_{w^*}-*USCO*, it suffices to show that $Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is $\rho_{w^* \times w^*}$-closed in $\mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y$. Let $\{(v^n, U^n_{(\cdot)})\}_n$ be any sequence in $Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ such that $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$ and $U^n_{(\cdot)} \underset{\rho_{w^*}}{\longrightarrow} U^*_{(\cdot)}$. We have then a subsequence, $\{(v^{n_k}, U^{n_k}_{(\cdot)})\}_k$, such that

$$v^{n_k} \underset{K}{\longrightarrow} \widehat{v} \text{ and } U^{n_k}_{(\cdot)} \underset{K}{\longrightarrow} \widehat{U}_{(\cdot)}, \text{ with } \widehat{v}(\omega) = v^*(\omega) \text{ and } \widehat{U}_{\omega} = U^*_{\omega} \text{ a.e. } [\mu].$$

Because $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a K-correspondence, $(\widehat{v}, \widehat{U}_{(\cdot)}) \in Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$. Thus, we have $(v^*, U^*_{(\cdot)}) \in Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$.

Next, for $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ a ρ_{w^*}-ρ_{w^*}-USCO, we will show that because the dominating probability measure, μ, is nonatomic, for each v, $\mathcal{S}^{\infty}(\mathcal{P}_v)$ is contractible.

As shown by Fryszkowski (1983), if μ is nonatomic, Lyapunov's Theorem (1940) on the range of a vector measure guarantees the existence of a family of measurable

[10] An *USCO* is an upper semicontinuous correspondence taking nonempty, compact values (e.g., Hola and Holy, 2015).

sets, $\{E_t\}_{t\in[0,1]}$, such that

$$\left.\begin{array}{c} t' \leq t \Rightarrow E_{t'} \subseteq E_t,\ E_0 = \varnothing \text{ and } E_1 = \Omega, \text{ and} \\ \mu(E_t) = t\mu(\Omega) = t. \end{array}\right\} \tag{45}$$

Using the properties of this system of measurable sets and the decomposability of $\mathcal{S}^\infty(\mathcal{P}_v)$ for each $v \in \mathcal{L}^\infty_Y$, we will show that for each v the function $h_v(\cdot,\cdot)$ given by

$$h_v(U_{(\cdot)}, t) := U^1_{(\cdot)} I_{E_t}(\cdot) + U_{(\cdot)} I_{\Omega\setminus E_t}(\cdot) \tag{46}$$

is a homotopy (i.e., $h_v(\cdot,\cdot) : \mathcal{S}^\infty(\mathcal{P}_v) \times [0,1] \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v)$ is jointly continuous in $(U_{(\cdot)}, t)$ on $\mathcal{S}^\infty(\mathcal{P}_v) \times [0,1]$ and is a contraction of $\mathcal{S}^\infty(\mathcal{P}_v)$ to $U^1_{(\cdot)}$). Here $v \in \mathcal{L}^\infty_Y$ is fixed, $I_E(\cdot)$ is the indicator function of set E, and $U^1_{(\cdot)}$ is any fixed selection in $\mathcal{S}^\infty(\mathcal{P}_v)$.

It suffices to show that $h_v(\cdot,\cdot)$ is $\rho_{w^*\times|\cdot|}$-ρ_{w^*}-continuous. Let $\{(U^n_{(\cdot)}, t^n)\}_n$ be a sequence such that

$$U^n_{(\cdot)} \underset{\rho_{w^*}}{\longrightarrow} U^*_{(\cdot)} \text{ and } t^n \underset{R}{\longrightarrow} t^*.$$

We must show that

$$h_v(U^n_{(\cdot)}, t^n) \underset{\rho_{w^*}}{\longrightarrow} h_v(U^*_{(\cdot)}, t^*) \in \mathcal{S}^\infty(\mathcal{P}_v). \tag{47}$$

Rewriting and substituting, we must show that for all $l \in \mathcal{L}^1_{R^m}$,

$$\left.\begin{array}{l} H = \underbrace{\int_\Omega \left\langle \left(U^1_\omega I_{E_{t^n}}(\omega) - U^1_\omega I_{E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(a)} \\ \quad + \underbrace{\int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(b)} \longrightarrow 0. \end{array}\right\} \tag{48}$$

We have for (48)(b) that

$$\underbrace{\int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(b)}$$

$$= \int_\Omega \left\langle U^n_\omega, l(\omega) I_{\Omega\setminus E_{t^n}}(\omega)\right\rangle d\mu(\omega) - \int_\Omega \left\langle U^*_\omega, l(\omega) I_{\Omega\setminus E_{t^*}}(\omega)\right\rangle d\mu(\omega).$$

Because the sequence $\{l(\cdot) I_{\Omega\setminus E_{t^n}}(\cdot)\}_n \subset \mathcal{L}^1_{R^m}$ $\|\cdot\|_1$-converges to $l(\cdot) I_{\Omega\setminus E_{t^*}}(\cdot) \in \mathcal{L}^1_{R^m}$, the fact that $U^n_{(\cdot)} \underset{\rho_{w^*}}{\longrightarrow} U^*_{(\cdot)}$ implies that

$$\int_\Omega \left\langle U^n_\omega, l(\omega) I_{\Omega\setminus E_{t^n}}(\omega)\right\rangle d\mu(\omega) \longrightarrow \int_\Omega \left\langle U^*_\omega, l(\omega) I_{\Omega\setminus E_{t^*}}(\omega)\right\rangle d\mu(\omega). \tag{49}$$

Thus, in expression (48), $(b) \longrightarrow 0$. Similarly, for (48)(a) we have that

$$\underbrace{\int_{\Omega} \left\langle \left(U^1_\omega I_{E_{t^n}}(\omega) - U^1_\omega I_{E_{t^*}}(\omega) \right), l(\omega) \right\rangle d\mu(\omega)}_{(a)}$$
$$= \int_{\Omega} \left\langle U^1_\omega, l(\omega) I_{E_{t^n}}(\omega) \right\rangle d\mu(\omega) - \int_{\Omega} \left\langle U^1_\omega, l(\omega) I_{E_{t^*}}(\omega) \right\rangle d\mu(\omega),$$

and because the sequence $\{l(\cdot)I_{E_{t^n}}(\cdot)\}_n \subset \mathcal{L}^1_{R^m}$ $\|\cdot\|_1$-converges to $l(\cdot)I_{E_{t^*}}(\cdot) \in \mathcal{L}^1_{R^m}$, we have that

$$\int_{\Omega} \left\langle U^1_\omega, l(\omega) I_{E_{t^n}}(\omega) \right\rangle d\mu(\omega) \longrightarrow \int_{\Omega} \left\langle U^1_\omega, l(\omega) I_{E_{t^*}}(\omega) \right\rangle d\mu(\omega). \tag{50}$$

We have then, in expression (48), $(a) \longrightarrow 0$.

Together, (49) and (50) imply that (47) holds. Thus, given the properties of the Lyapunov system (45), the function given in (46) is, for each $v \in \mathcal{L}^\infty_Y$, $\rho_{w^* \times |\cdot|}$-$\rho_{w^*}$-continuous, and therefore, specifies a homotopy for the set of measurable selections, $\mathcal{S}^\infty(\mathcal{P}_v)$ - and thus for each v, $\mathcal{S}^\infty(\mathcal{P}_v)$ is contractible. **Q.E.D.**

Our proof that $\mathcal{S}^\infty(\mathcal{P}_v)$ is contractible for each v is a modified version of the proof given by Mariconda (1992) showing that if the underlying probability space is nonatomic then any decomposable subset of E-valued, Bochner integrable functions in $\mathcal{L}^1_E$ is contractible (where E is a Banach space). In Mariconda's result, the space of functions is equipped with the norm in $\mathcal{L}^1_E$, while here our space of functions (with each function taking values in $Y \subset R^m$) is equipped with the metric, ρ_{w^*}, compatible with the w^* topology.

6.2 The Approximability and Fixed Point Results

The importance of the K-limit property in a nonatomic probability space derives from the fact that it guarantees that $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is *a* ρ_{w^*}-ρ_{w^*}-USCO taking contractible values. This in turn guarantees approximability and the existence of fixed points, as our next results show.

Theorem 3 *($\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is ρ_{w^*}-ρ_{w^*}-approximable)*
Let

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := U(\omega, v, \mathcal{N}(\omega, v))$$

be the Nash payoff correspondence induced by a Nash correspondence belonging to a nonatomic convex DSG. Then $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is a ρ_{w^*}-ρ_{w^*}-approximable.

Proof: Because $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is a contractibly-valued ρ_{w^*}-ρ_{w^*}-USCO, by Corollary 5.6 in Gorniewicz, Granas, and Kryszewski (1991), because $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is defined on the ANR (absolute neighborhood retract) space of value functions $\mathcal{L}^\infty_Y$ and takes nonempty, compact, and contractible values in $\mathcal{L}^\infty_Y$ (and hence ∞-proximally connected values - see Theorem 5.3 in Gorniewicz, Granas, and Kryszewski, 1991), $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is a J mapping. Therefore, by Theorem 5.12 in Gorniewicz, Granas, and Kryszewski (1991), $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is ρ_{w^*}-ρ_{w^*}-approximable. **Q.E.D.**

We can now state our main fixed point result.

Theorem 4 *($\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ has fixed points)*
Let

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := U(\omega, v, \mathcal{N}(\omega, v))$$

be the Nash payoff correspondence induced by a Nash correspondence belonging to a nonatomic convex DSG. Then $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ has a fixed point (i.e., there exists $v^* \in \mathcal{L}^{\infty}_Y$ such that $v^* \in \mathcal{S}^{\infty}(\mathcal{P}_{v^*})$).

Proof: By Theorem 3 above $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is ρ_{w^*}-ρ_{w^*}-approximable. Therefore, we have for each n, a ρ_{w^*}-ρ_{w^*}-continuous function, $g^n(\cdot) : \mathcal{L}^{\infty}_Y \longrightarrow \mathcal{L}^{\infty}_Y$, such that for each $(v^n, U^n_{(\cdot)}) \in Grg^n \subset \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y$ (i.e., for each $(v^n, U^n_{(\cdot)}) \in \mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y$, with $U^n_{(\cdot)} = g^n(v^n) \in \mathcal{L}^{\infty}_Y$) there exists $(\overline{v}^n, \overline{U}^n_{(\cdot)}) \in Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ such that

$$\rho_{w^*}(v^n, \overline{v}^n) + \rho_{w^*}(U^n_{(\cdot)}, \overline{U}^n_{(\cdot)}) < \tfrac{1}{n}. \tag{51}$$

Equivalently, for any positive integer, n, $Grg^n \subset B_{w^* \times w^*}(\frac{1}{n}, Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}))$. Thus, the graph of the continuous function $g^n : \mathcal{L}^{\infty}_Y \longrightarrow \mathcal{L}^{\infty}_Y$ is contained in the $\rho_{w^* \times w^*}$-open ball of radius $\frac{1}{n}$ about the graph of $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$.

Because each of the functions, g^n, is ρ_{w^*}-ρ_{w^*}-continuous and defined on the ρ_{w^*}-compact and convex subset, $\mathcal{L}^{\infty}_Y$, in $\mathcal{L}^{\infty}_{R^m}$, taking values in $\mathcal{L}^{\infty}_Y$, it follows from the fixed point theorem of Schauder (see Aliprantis and Border, 2006), that each g^n has a fixed point, $v^n \in \mathcal{L}^{\infty}_Y$ (i.e., for each n there exists some $v^n \in \mathcal{L}^{\infty}_Y$ such that $v^n = g^n(v^n)$). Let $\{v^n\}_n$ be a fixed point sequence corresponding to the sequence of ρ_{w^*}-ρ_{w^*}-continuous approximating functions, $\{g^n(\cdot)\}_n$. Expression (51) can now be rewritten as follows: for each v^n in the fixed point sequence, there is a corresponding pair, $(\overline{v}^n, \overline{U}^n_{(\cdot)}) \in Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, such that

$$\rho_{w^*}(v^n, \overline{v}^n) + \rho_{w^*}(g^n(v^n), \overline{U}^n_{(\cdot)}) < \tfrac{1}{n},$$

and therefore such that

$$\underbrace{\rho_{w^*}(v^n, \overline{v}^n)}_{A} + \underbrace{\rho_{w^*}(v^n, \overline{U}^n_{(\cdot)})}_{B} < \tfrac{1}{n}. \tag{52}$$

By the ρ_{w^*}-compactness of $\mathcal{L}^{\infty}_Y$, we can assume WLOG that the fixed point sequence, $\{v^n\}_n \subset \mathcal{L}^{\infty}_Y$, ρ_{w^*}-converges to a limit $v^* \in \mathcal{L}^{\infty}_Y$. Thus, by part A of (52), as $n \longrightarrow \infty$ we have

$$v^n \underset{\rho_{w^*}}{\longrightarrow} v^* \text{ and } \overline{v}^n \underset{\rho_{w^*}}{\longrightarrow} v^*,$$

and therefore by part B of (52), as $n \longrightarrow \infty$ we have

$$\overline{U}^n_{(\cdot)} \underset{\rho_{w^*}}{\longrightarrow} v^*.$$

Because $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is $\rho_{w^* \times w^*}$-closed in $\mathcal{L}^{\infty}_Y \times \mathcal{L}^{\infty}_Y$,

$$\{(\overline{v}^n, \overline{U}^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}),$$

and $\overline{v}^n \underset{\rho_{w^*}}{\longrightarrow} v^*$ and $\overline{U}^n_{(\cdot)} \underset{\rho_{w^*}}{\longrightarrow} v^*$ imply that $(v^*, v^*) \in \mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$. Therefore, $v^* \in \mathcal{S}^{\infty}(\mathcal{P}_{v^*})$. **Q.E.D.**

7 The Error In Our Earlier Paper and Its Correction

In our earlier paper, we considered uC correspondences, $\mathcal{N}(\cdot,\cdot)$, having continuum-valued uC sub-correspondences, $\eta(\cdot,\cdot)$, i.e., continuum-valued uC correspondences such that

$$Gr\eta(\omega,\cdot) \subseteq Gr\mathcal{N}(\omega,\cdot) \text{ a.e. } [\mu]. \tag{53}$$

We will denote by $\mathcal{UC}^{\mathcal{N}}_{C_f(X)}$ the set of all continuum-valued uC sub-correspondences belonging to the uC correspondence, $\mathcal{N}(\cdot,\cdot)$. Here $C_f(X)$ denotes the collection of all nonempty, closed (and hence, compact), connected subsets of X - i.e., the collection of all sub-continua belonging to X). Noting that under assumptions $[OSG]$, if $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}_{C_f(X)}$, then for each $d = 1, 2, \ldots, m$,

$$(\omega, v) \longrightarrow p_d(\omega, v) := U_d(\omega, v_d, \eta(\omega, v)), \tag{54}$$

player $d's$ Nash payoff sub-correspondence, $p_d(\cdot,\cdot)$, is interval-valued, hence contractibly-valued. We then showed that there exists $v^* \in \mathcal{L}^\infty_Y$ such that

$$v^*(\omega) \in p_1(\omega, v^*) \times \cdots \times p_1(\omega, v^*) \text{ a.e. } [\mu]. \tag{55}$$

This is all correct - but this is not what we intended to show, nor does (55) allow us to concluded that all convex $DSGs$ have stationary Markov perfect equilibria. Our objective was to show that there exists $v^* \in \mathcal{L}^\infty_Y$ such that

$$v^*(\omega) \in p(\omega, v^*) := \{(U_1(\omega, v_1^*, x), \ldots, U_m((\omega, v_m^*, x)) : x \in \eta(\omega, v^*)\} \text{ a.e. } [\mu]. \tag{56}$$

But we incorrectly stated that (56) could be deduced from (55) using implicit measurable selection methods (e.g., Theorem 7.1 in Himmelberg, 1975). This is *not* the case. We note that $p(\omega, v^*)$ is a subset of $p_1(\omega, v^*) \times \cdots \times p_1(\omega, v^*)$. Therefore, a $v^* \in \mathcal{L}^\infty_Y$ satisfying (**??**) does not necessarily satisfy (56). Moreover, in order to conclude that all convex $DSGs$ have stationary Markov perfect equilibria (and in this case nonatomic, convex $DSGs$), we must be able to show that there exists $v^* \in \mathcal{L}^\infty_Y$ satisfying (56). Here we have corrected our earlier paper, and proved the fixed point result we intended to prove. We have shown that under assumptions $[OSG]$ there exists $v^* \in \mathcal{L}^\infty_Y$ satisfying (56).

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