THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

Economic and Social Research Council

Parameterized State-Contingent Games, 3M Minimal Nash Correspondences, and Connectedness

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SRC Discussion Paper No 116
March 2022

Systemic Risk Centre

Discussion Paper Series

ISSN 2054-538X

Abstract
Under mild assumptions on primitives, we show that all parameterized state-contingent games (PSGs) have upper Caratheodory (uC) Nash (equilibrium) correspondences which contain minimal uC Nash correspondences having the 3M property (defined here). This implies that all PSGs have Nash correspondences made up of minimal uC Nash correspondences taking closed, connected, and essential Nash equilibrium values (essential in the sense of Fort, 1950). It then follows from Fu and Page (2022b), that because all PSGs have continuum valued minimal Nash correspondences, all PSGs have Caratheodory approximable Nash payoff correspondences - which in turn implies that all PSGs have approximable Nash payoff selection correspondences, and therefore have Nash payoff selection correspondences with fixed points.

Keywords: m-tuples of Caratheodory functions, upper Caratheodory correspondences, continuum valued upper Caratheodory sub-correspondences, weak star upper semicontinuous measurable selection valued correspondences, approximate Caratheodory selections, fixed points of nonconvex, measurable selection valued correspondences induced by the composition of an m-tuple of Caratheodory functions with a continuum valued upper Caratheodory sub-correspondence.

This paper is published as part of the Systemic Risk Centre's Discussion Paper Series. The support of the Economic and Social Research Council (ESRC) in funding the SRC is gratefully acknowledged [grant number ES/R009724/1].

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Published by
Systemic Risk Centre
The London School of Economics and Political Science
Houghton Street
London WC2A 2AE

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Parameterized State-Contingent Games, $3M$ Minimal Nash Correspondences, and Connectedness

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April 3, 2022[3] [4]

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[3]Both authors thank Ann Law, J. P. Zigrand and Jon Danielsson for all their support and hospitality during many visits to the Systemic Risk Centre at the London School of Economics during the summers from 2014 until the present. Page acknowledges financial support from the Systemic Risk Centre (under ESRC grant numbers ES/K002309/1 and ES/R009724/1). Fu thanks JSPS KAKENHI for financial support under grant number 19K13662. 3M and Connectedness_jf-fp_4–3-22

[4]This paper is a greatly revised and extended version of, "Parameterized Games, Minimal Nash Correspondences, and Connectedness," Systemic Risk Centre DP 45, London School of Economics and Political Science, September 2015 (Page 2015b)

Abstract

Under mild assumptions on primitives, we show that all parameterized state-contingent games ($\mathcal{PSG}s$) have upper Caratheodory (uC) Nash (equilibrium) correspondences which contain minimal uC Nash correspondences having the $3M$ property (defined here). This implies that all $\mathcal{PSG}s$ have Nash correspondences made up of minimal uC Nash correspondences taking closed, connected, and essential Nash equilibrium values (essential in the sense of Fort, 1950). It then follows from Fu and Page (2022b), that because all $\mathcal{PSG}$s have continuum valued minimal Nash correspondences, all $\mathcal{PSG}$s have Caratheodory approximable Nash payoff correspondences - which in turn implies that all $\mathcal{PSG}$s have Nash payoff selection correspondence with fixed points.

Key Words: m-tuples of real valued Caratheodory functions, upper Caratheodory correspondences, continuum valued upper Caratheodory sub-correspondences, weak star upper semicontinuous measurable selection valued sub-correspondences, approximate Caratheodory selections, fixed points of nonconvex, measurable selection valued correspondences induced by the composition of an m-tuple of Caratheodory functions with a continuum valued upper Caratheodory sub-correspondence.

1 Introduction

Economics and game theory are replete with examples of parameterized, state-contingent games ($\mathcal{PSG}s$). One of the most interesting examples can be found in the theory of discounted stochastic games ($\mathcal{DSG}s$) with uncountable state spaces and compact metric action spaces. We know that the key to showing that a discounted stochastic game has stationary Markov perfect equilibria ($SMPE$) is to show that the $\mathcal{PSG}$ underlying the discounted stochastic game contains an equilibrium state-contingent, one-shot game - that is, a one shot-game parameterized by an equilibrium vector of state-contingent prices - prices that players use to value their continued use of a particular strategy.[1] Once an equilibrium one-shot game has been found, the equilibrium stationary Markov perfect strategy profile is gotten by measurably stringing together, state-by-state, the Nash equilibria of the one-shot games corresponding to the equilibrium vector of state-contingent prices (i.e., the profile of equilibrium valuation functions). The hard problem is finding the equilibrium vector of valuation functions. This problem is a fixed point problem involving the measurable selection valued *Nash payoff selection correspondence*. We know that a vector of valuation functions is an equilibrium vector if and only if the vector is a fixed point of the Nash payoff selection correspondence. But this fixed point problem is very difficult because Nash payoff selection correspondences are, in general, neither convex valued nor closed valued in the appropriate topology (in this case the weak star topology). However, as was shown by Fu and Page (2022a) the problem can be solved by approximate fixed point methods provided the underlying upper Caratheodory (uC) Nash payoff correspondence contains a contractible-valued uC sub-correspondence. We show here that this will be the case if the underlying uC Nash correspondence contains a *uC sub-correspondence taking connected values* in the set of Nash equilibria. Here, we show that all $\mathcal{PSG}s$ satisfying the usual assumptions (i.e., the usual assumptions made in specifying a $\mathcal{DSG}$) have uC Nash correspondences containing minimal uC Nash correspondences taking closed connected values. In fact, we show here that a $\mathcal{PSG}'s$ Nash correspondence consists of state-contingent bundles of upper semicontinuous strands of connected Nash equilibria - implying that each player's Nash payoff correspondence consists of state-contingent bundles of upper semicontinuous strands of closed intervals of Nash payoffs. Thus, under the usual assumptions, in a $\mathcal{PSG}$, each player's uC Nash payoff correspondence is given by a bundle of uC sub-correspondences each taking closed interval values of Nash payoffs. As a consequence, players' uC Nash payoff sub-correspondences are Caratheodory approximable (see Kucia and Nowak, 2000), implying that the $\mathcal{PSG}'s$ induced Nash payoff selection correspondence has fixed points (see Fu and Page, 2022a).

Why do all $\mathcal{PSG}s$ satisfying the usual assumptions have minimal uC Nash sub-correspondences taking continuum values? We show that this is true for two reason: (1) because all $\mathcal{PSG}s$ have *Ky Fan correspondences* containing minimal Ky Fan sub-correspondences having the $3M$ property (defined below) implying that these minimal Ky Fan sub-correspondences take minimally essential, connected Nash equilibria values; and (2) because each minimal uC Nash correspondence belonging to a $\mathcal{PSG}$ is given by the *composition* of a minimal Ky Fan sub-correspondence with the *$\mathcal{PSG}'s$ collective security function*, a Caratheodory function, mapping from state-value function profile pairs into the $\mathcal{PSG}'s$ Ky Fan sets, where the Ky Fan values taken by the collective security function at each state-value function profile pair is determined by the $\mathcal{PSG}'s$ Nikaido-Isoda function. Our main contributions are to establish (1) and (2) above. The fixed point implications of what we do here are proved in Fu and Page (2022a), while the implications for the existence of SMPE in $\mathcal{DSG}s$ are established in Fu and Page (2022b).

[1] A $\mathcal{PSG}$ is a parameterized *collection* of m-player, state-contingent, strategic form games. Thus, the collection of one-shot games underlying a $\mathcal{DSG}$ is an example of a $\mathcal{PSG}$.

2 Parameterized, State-Contingent Games

2.1 Primitives, Assumptions, and Comments

2.1.1 Primitives

We begin by considering a collection of strategic form games,

$$\mathcal{G}(\Omega \times V) := \left\{ \mathcal{G}_{(\omega,v)} : (\omega, v) \in \Omega \times V \right\}, \tag{1}$$

where each (ω, v)-game is given by

$$\mathcal{G}_{(\omega,v)} := \left\{ \underbrace{\Phi_d(\omega, v_d)}_{\text{feasible actions}} , \underbrace{u_d(\omega, v_d, (\cdot,\cdot))}_{\text{payoff function}} \right\}_{d \in D} . \tag{2}$$

We will refer to the entire collection of (ω, v)-games as a *parameterized state-contingent game* (a $\mathcal{PSG}$). In a (ω, v)-game each player $d = 1, 2, \ldots, m$, seeks to choose a feasible action, $x_d \in \Phi_d(\omega, v_d)$ so as to maximize $d's$ payoff - i.e., so as to solve the problem

$$\max_{x_d \in \Phi_d(\omega, v_d)} u_d(\omega, v_d, x_d, x_{-d}),$$

given state ω, parameter value v_d, and the feasible actions, x_{-d}, chosen by the other players. Each (ω, v)-game is played by $m = |D|$ players with underlying primitives,

$$((\Omega, B_\Omega, \mu), V, \{X_d, \Phi_d(\cdot,\cdot), u_d(\cdot,\cdot,(\cdot,\cdot))\}_{d \in D}) . \tag{3}$$

2.1.2 Assumptions

We will maintain the following assumptions about the underlying primitives specifying the parameterized state-contingent games given in expression (3). We will refer to these assumptions collectively as assumptions [$\mathcal{PSG}$]:

(1) $D := \{1, 2, \ldots, m\}$ is the set of players with typical element d.
(2) (Ω, B_Ω, μ) is a probability space of states with typical element, ω, where Ω is complete separable (Polish) metric space and μ is a probability measure defined on the Borel σ-field B_Ω.[2]
(3) X_d is a ***convex****, compact metrizable subset of a locally convex Hausdorff topological vector space E_d and is equipped with a metric, ρ_{X_d}, compatible with the locally convex topology inherited from E_d.*
(4) $X := X_1 \times \cdots \times X_m$ is the product space of action profiles equipped with the sum metric, $\rho_X := \sum_d \rho_{X_d}$. as well as the Borel product σ-field, $B_X = B_{X_1} \times \cdots \times B_{X_m}$, generated by the ρ_X-open sets in X.
(5) $Y := Y_1 \times \cdots \times Y_m$ is the product space of player payoff profiles, $u := (u_1, \ldots, u_m)$, such that for each player d, $Y_d := [-M, M]$, $M > 0$ and Y_d is equipped with the absolute value metric,

$$\rho_{Y_d}(u_d^0, u_d^1) := \left| u_d^0 - u_d^1 \right|, \tag{4}$$

and $Y := Y_1 \times \cdots \times Y_m$ is equipped with the sum metric, $\rho_Y := \sum_{d=1}^m \rho_{Y_d}$, as well as the Borel product σ-field, $B_Y = B_{Y_1} \times \cdots \times B_{Y_m}$, generated by the ρ_Y-open sets in Y.

[2] Note that the σ-field, B_Ω is countably generated. All the results we present here remain valid if instead we assume that Ω is an abstract set, but one equipped with a countably generated σ-field (see Ash, 1972).

(5) V_d is player $d's$ parameter space given by a locally connected continuum of parameters equipped with M-convex metric ρ_V.
(8) $V := V_1 \times \cdots \times V_m$ is the set of player parameter profiles, $v := (v_1, \ldots, v_m)$ equipped with the M-convex sum metric, $\rho_V := \sum_d \rho_{V_d}$, as well as the Borel product σ-field, $B_V = B_{V_1} \times \cdots \times B_{V_m}$, generated by the ρ_V-open sets in V.
(9) $\Phi_d(\cdot,\cdot)$ is player $d's$ action choice constraint correspondence, a Caratheodory set-valued mapping (i.e., measurable in ω and continuous in v_d - i.e., both upper semicontinuous and lower semicontinuous in v_d),

$$\Phi_d(\cdot,\cdot) : \Omega \times V_d \longrightarrow P_{fc}(X_d), \tag{5}$$

defined on the space of state-parameter pairs, $\Omega \times V_d$, taking nonempty, ρ_{X_d}-closed (and hence ρ_{X_d}-compact), convex values in player $d's$ action choice set X_d (where $P_{fc}(X_d)$ denotes the collection of all nonempty, ρ_{X_d}-closed and classically convex subsets of X_d).
(10) $\Phi(\cdot,\cdot) := \Phi_1(\cdot,\cdot) \times \cdots \times \Phi_m(\cdot,\cdot)$ is players' action profile constraint correspondence, a Caratheodory set-valued mapping defined on the space of state-parameter pairs, $\Omega \times V$, taking nonempty, ρ_X-closed (and hence ρ_X-compact) convex values in players' action profile set X.[3]
(11) $u_d(\cdot,\cdot,(\cdot,\cdot))$ is player $d's$ payoff function,

$$(\omega, v_d, (x_d, x_{-d})) \longrightarrow u_d(\omega, v_d, (x_d, x_{-d})) \in Y_d := [-M, M], \tag{6}$$

a (B_Ω, B_{Y_d})- measurable function on Ω for each $(v_d, (x_d, x_{-d})) \in V_d \times (X_d \times X_{-d}) := V_d \times X$, and a jointly continuous function on $V_d \times (X_d \times X_{-d})$ for each $\omega \in \Omega$, with

$$x_d \longrightarrow u_d(\omega, v_d, (x_d, x_{-d}))$$

concave in x_d on X_d for each $(\omega, v_d, x_{-d}) \in \Omega \times V_d \times X_{-d}$.[4]
(12) $\varphi(\cdot,\cdot,(\cdot,\cdot))$ is the game's Nikaido-Isoda function

$$\left.\begin{array}{c} y \longrightarrow \varphi(\omega, v, (y, x)) \\ := \sum_{d=1}^m u_d(\omega, v_d, (y_d, x_{-d})) - \sum_{d=1}^m u_d(\omega, v_d, (x_d, x_{-d})), \end{array}\right\} \tag{7}$$

a function concave in $y := (y_d, y_{-d})$ on $X = X_d \times X_{-d}$ for each $x := (x_d, x_{-d}) \in X$.

2.1.3 Comments on Primitives and Assumptions

(1) Because player $d's$ compact metric space of parameters, V_d, is connected and locally connected, V_d is a Peano continuum, and therefore can be equipped with an M-convex metric ρ_{V_d}. (see Illanes and Nadler, 1999).[5] A metric space, (V_d, ρ_{V_d}), is M-convex (M refers for Menger, 1928) if for any two distinct elements, v_d^0 and v_d^1 in V_d there is a third element, $\overline{v}_d$, in V_d such that

$$\rho_{V_d}(v_d^0, v_d^1) = \rho_{V_d}(v_d^0, \overline{v}_d) + \rho_{V_d}(\overline{v}_d, v_d^1). \tag{8}$$

[3] See Himmelberg (1975) and Wagner (1977) for extensive discussions of continuity and measurability issues related to correspondences.

[4] If the function

$$x_d \longrightarrow u_d(\omega, v_d, (x_d, x_{-d}))$$

is concave in x_d then for each $\lambda \in [0,1]$,

$$u_d(\omega, v_d, \lambda x_d^1 + (1-\lambda)x_d^2, x_{-d}) \geq \lambda u_d(\omega, v_d, x_d^1, x_{-d}) + (1-\lambda) u_d(\omega, v_d, x_d^2, x_{-d}).$$

[5] Conversely, any compact, M-convex metric space is a Peano continuum.

Because any compact metric space, (Z, d_Z), that is a Peano continuum (connected and locally connected) has an equivalent M-convex metric, ρ_Z, without loss of generality, we can assume that each of our Peano continua, V_d, is equipped with an M-convex metric ρ_{V_d}. Moreover, if we equip the product space, $V := V_1 \times \cdots \times V_m$, with the sum metric $\rho_V := \sum_d \rho_{V_d}$, then (V, ρ_V) is a compact, M-convex metric space, and therefore a Peano continuum (see III.10 in Illanes and Nadler, 1999, and Theorem 1.2 in Goodykoontz and Nadler, 1982). It should be emphasized that *local connectedness* of the parameter space, in this case V, is required in order to show that the $3M$ property implies (and is implied by) the connectedness of the values of the minimal uC Nash correspondence.

(2) Suppose (Z, ρ_Z) is a compact metric space. Consider the hyperspace of nonempty, ρ_Z-closed subsets $P_f(Z)$. The distance from a point $z \in Z$ to a set $C \in P_f(Z)$ is given by

$$dist_{\rho_Z}(z, C) := \inf_{z' \in C} \rho_Z(z, z'). \tag{9}$$

Given two sets B and C in 2^Z, the excess of B over C is given by

$$e_{\rho_Z}(B, C) := \sup_{z \in B} dist_{\rho_Z}(z, C). \tag{10}$$

Given two sets B and C in $P_f(Z)$, the Hausdorff distance in $P_f(Z)$ between B and C is given by

$$h_{\rho_Z}(B, C) = \max\{e_{\rho_Z}(B, C), e_{\rho_Z}(C, B)\}. \tag{11}$$

Often we will write h rather than h_{ρ_Z} - when the underlying metric is clear.

Finally, let $P_f(V)$ be the hyperspace of *nonempty*, closed (and hence compact) subsets of V, equipped with the Hausdorff metric h_{ρ_V} induced by the M-convex metric, ρ_V. Because the metric space, (V, ρ_V), is compact and M-convex, the metric hyperspace $(P_f(V), h_{\rho_V})$ is compact and M-convex (see Duda, 1970).

2.2 Correspondences

2.2.1 Upper Caratheodory (uC) Correspondences

Consider an upper Caratheodory (uC) correspondence,

$$\mathcal{N}(\cdot, \cdot) : \Omega \times V \longrightarrow P_f(X), \tag{12}$$

jointly measurable in (ω, v) and upper semicontinuous in v, taking nonempty ρ_X-closed (and hence, ρ_X-compact) values in X. The uC correspondence, $\mathcal{N}(\cdot, \cdot)$, has graph

$$Gr\mathcal{N}(\cdot, \cdot) := Gr\mathcal{N} := \{(\omega, v, x) \in \Omega \times V \times X : x \in \mathcal{N}(\omega, v)\}. \tag{13}$$

Given ω or v, for $S \subset X$ define,

$$\left.\begin{array}{c} \mathcal{N}_\omega^-(S) := \{v \in V : \mathcal{N}_\omega(v) \cap S \neq \varnothing\}, \\ \text{and} \\ \mathcal{N}_v^-(S) := \{\omega \in \Omega : \mathcal{N}_v(\omega) \cap S \neq \varnothing\}, \end{array}\right\} \tag{14}$$

where for fixed ω, $\mathcal{N}_\omega(\cdot) := \mathcal{N}(\omega, \cdot)$, and for fixed v, $\mathcal{N}_v(\cdot) := \mathcal{N}(\cdot, v)$. Finally, let

$$\mathcal{N}^-(S) := \{(\omega, v) \in \Omega \times V : \mathcal{N}(\omega, v) \cap S \neq \varnothing\}. \tag{15}$$

We have the following definitions (see Wagner, 1977). Given correspondence, $\mathcal{N}(\cdot, \cdot)$, we say that,

(a) $\mathcal{N}_v(\cdot)$ is weakly measurable (or measurable) if for all S open in X, $\mathcal{N}_v^-(S) \in B_\Omega$;

(b) $\mathcal{N}_\omega(\cdot)$ is upper semicontinuous if for all S closed X, $\mathcal{N}_\omega^-(S)$ is closed in V;

(c) $\mathcal{N}(\cdot,\cdot)$ is product measurable (i.e., jointly measurable in ω and v) if for all S open in X, $\mathcal{N}^-(S) \in B_\Omega \times B_V$.

(d) $\mathcal{N}(\cdot,\cdot)$ is upper Caratheodory if $\mathcal{N}(\cdot,\cdot)$ is product measurable and for each ω, $\mathcal{N}_\omega(\cdot)$ is upper semicontinuous.

Because X is a compact metric space and $\mathcal{N}_v(\cdot)$ is closed valued, weak measurability of $\mathcal{N}_v(\cdot)$ implies that for each v $\mathcal{N}_v^-(S) \in B_\Omega$ for S closed in X.

We will denote by,

$$\mathcal{N}^{USCO} := \{\mathcal{N}(\omega,\cdot) : \omega \in \Omega\}, \tag{16}$$

the nonempty, ρ_X-compact valued, upper semicontinuous part of $\mathcal{N}(\cdot,\cdot)$. Following the terminology of Hola and Holy (2015), we will refer to $\mathcal{N}^{USCO}$ as the USCO part of $\mathcal{N}$. We will denote the measurable part of $\mathcal{N}(\cdot,\cdot)$, by

$$\mathcal{N}^{B_\Omega} := \{\mathcal{N}(\cdot,v) : v \in V\}. \tag{17}$$

For each state-parameter pair, (ω, v), the graphs of the USCO part and the measurable part are given by,

$$\left.\begin{array}{c} Gr\mathcal{N}(\omega,\cdot) := Gr\mathcal{N}_\omega := \{(v,x) \in V \times X : x \in \mathcal{N}(\omega,v)\}, \\ \text{and} \\ Gr\mathcal{N}(\cdot,v) := Gr\mathcal{N}_v := \{(\omega,x) \in \Omega \times X : x \in \mathcal{N}(\omega,v)\}. \end{array}\right\} \tag{18}$$

Moreover, by Lemma 3.1(ii) in Kucia and Nowak (2000), the correspondence, $\omega \longrightarrow Gr\mathcal{N}_\omega$, is measurable.

We will denote the collection of all upper Caratheodory correspondences defined on $\Omega \times V$ with nonempty, compact values in X by $\mathcal{UC}_{\Omega\times V\text{-}P_f(X)}$.

2.2.2 USCOs

For compact metric spaces (V, ρ_V) and (X, ρ_X), let $\mathcal{U}_{V-P_f(X)} := \mathcal{U}(V, P_f(X))$ denote the collection of all upper semicontinuous correspondences taking nonempty, ρ_X-closed (and hence ρ_X-compact) values in X. Following the literature, we will call such mappings, USCOs (see Crannell, Franz, and LeMasurier, 2005, Anguelov and Kalenda, 2009, and Hola and Holy, 2009). We know that $\Psi \in \mathcal{U}_{V\text{-}P_f(X)}$ if and only if $Gr\Psi$ is compact, where

$$Gr\Psi := \{(v,x) \in V \times X : x \in \Psi(v)\}.$$

Given $\Psi \in \mathcal{U}_{V\text{-}P_f(X)}$ denote by $\mathcal{U}_{V\text{-}P_f(X)}[\Psi]$ the collection of all sub-USCOs belonging to Ψ, that is, all USCOs $\phi \in \mathcal{U}_{V\text{-}P_f(X)}$ whose graph,

$$Gr\phi := \{(v,x) \in V \times X : x \in \phi(v)\},$$

is contained in the graph of Ψ, $Gr\Psi$. We will call any sub-USCO, $\phi \in \mathcal{U}_{V\text{-}P_f(X)}[\Psi]$ a minimal USCO belonging to Ψ, if for any other sub-USCO, $\psi \in \mathcal{U}_{V\text{-}P_f(X)}[\Psi]$, $Gr\psi \subseteq Gr\phi$ implies that $Gr\psi = Gr\phi$. We will use the special notation, $[\Psi]$, to denote the collection of all minimal USCOs belonging to Ψ. We know that for any USCO Ψ, $[\Psi] \neq \varnothing$ (see Drewnowski and Labuda, 1990). In general, we say that ψ is a minimal USCO, if for any other USCO $\phi \in \mathcal{U}_{V-P_f(X)}$, $Gr\phi \subseteq Gr\psi$ implies that $Gr\phi = Gr\psi$. Let $\mathcal{M}_{V\text{-}P_f(X)}$ denote the collection of all minimal USCOs. The following characterizations of minimal USCOs (gathered from Anguelov and Kalenda, 2009, and Hola and Holy, 2009) will be useful later.

Characterizations of Minimal USCOs (Anguelov and Kalenda, 2009, and Hola and Holy, 2009)
Suppose assumptions [$\mathcal{PSG}$] hold. The following statements are equivalent:
(1) $\eta(\cdot) \in \mathcal{U}_{V-P_f(X)}$ *is a minimal USCO.*
(2) If $U \subset V$ *and* $V \subset X$ *are open sets such that* $\eta(U) \cap V \neq \emptyset$*, then there is a nonempty open subset* W *of* U *such that* $\eta(W) \subset V$*.*
(3) If $U \subset V$ *is an open set and* $F \subset X$ *is a closed set such that* $\eta(v) \cap F \neq \emptyset$ *for each* $v \in U$*, then* $\eta(U) \subset F$*.*
(4) There exists a quasi-continuous selection f *of* $\eta(\cdot)$ *such that* $\overline{Grf} = Gr\eta$*.*[6]
(5) Every selection f *of* $\eta(\cdot)$ *is quasi-continuous and* $\overline{Grf} = Gr\eta$*.*[7]

Finally, we say that an USCO, $\Psi \in \mathcal{U}_{V\text{-}P_f(X)}$, is quasi-minimal if for some $\psi \in \mathcal{U}_{V\text{-}P_f(X)}$, $[\Psi] = \{\psi\}$ (i.e., Ψ has one and only one minimal USCO). Let $\mathcal{Q}_{V\text{-}P_f(X)}$ denote the collection of all quasi-minimal USCOs. We will denote by

$$S_\Psi := \{v \in V : \Psi(v) \text{ is a singleton}\}, \tag{19}$$

the the subset where Ψ takes singleton values. Under our primitives and assumptions, if $\Psi \in \mathcal{Q}_{V\text{-}P_f(X)}$, then by Lemma 7 in Anguelov and Kalenda (2009), S_Ψ is a residual set - and in particular, a G_δ set ρ_V-dense in V.

2.2.3 Minimal uC Correspondences and Minimal USCOs

For $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times V\text{-}P_f(X)}$, let

$$\mathcal{UC}^{\mathcal{N}} := \mathcal{UC}_{\Omega\times V\text{-}P_f(X)}[\mathcal{N}(\cdot,\cdot)] \tag{20}$$

denote the collection of all upper Caratheodory mappings, $\eta(\cdot,\cdot)$, belonging to $\mathcal{N}(\cdot,\cdot)$. Thus, $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$ if and only if $\eta(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times V\text{-}P_f(X)}$ and

$$Gr\eta(\omega,\cdot) \subset Gr\mathcal{N}(\omega,\cdot) \text{ for all } \omega.$$

We will be interested in *uC* sub-correspondences, $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$, with the property that for each ω, $\eta(\omega,\cdot)$ is a *minimal USCO belonging to* $\mathcal{N}(\omega,\cdot)$. Already we know that (i) for each ω, $[\mathcal{N}(\omega,\cdot)]$ is nonempty (e.g., see Drewnowski and Labuda, 1990), and (ii) for each ω and $\eta(\omega,\cdot) \in [\mathcal{N}(\omega,\cdot)]$, $S_{\eta(\omega,\cdot)}$ is a G_δ set ρ_V-dense in V, and therefore, for each $(\omega, v) \in \omega \times S_{\eta(\omega,\cdot)}$, $\eta(\omega, v)$ is single-valued. What we don't know is whether or not the *uC* correspondence, $\mathcal{N}(\cdot,\cdot)$, contains a *uC* sub-correspondence, $\eta(\cdot,\cdot)$, such that for each ω, $\eta(\omega,\cdot)$ is a minimal USCO belonging to $\mathcal{N}(\omega,\cdot)$. We call any such *uC* sub-correspondence a minimal *uC* sub-correspondence, and we denote by,

$$\mathcal{MUC}^{\mathcal{N}} := \left\{\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}} : \eta(\omega,\cdot) \in [\mathcal{N}(\omega,\cdot)] \text{ for all } \omega\right\}, \tag{21}$$

the collection of all minimal *uC* sub-correspondences belonging to $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times V\text{-}P_f(X)}$.

Our first result shows that for any $\mathcal{N} \in \mathcal{UC}_{\Omega\times V\text{-}P_f(X)}$, $\mathcal{MUC}^{\mathcal{N}}$ is nonempty.

[6] A function $f^* : Z \to X$ is quasicontinuous at z^0 if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that inside the open ball, $B_{\rho_Z}(\delta, z^0)$, there is contained an open set, U, such that for all $z \in U$,

$$f^*(z) \in B_{\rho_X}(\varepsilon, f^*(z^0)).$$

[7] Note that if a function is continuous, it is automatically quasi-continuous.

Theorem 1 *(Nonemptiness of $\mathcal{MUC}^{\mathcal{N}}$)*
Suppose assumptions [$\mathcal{PSG}$] hold. For any upper Caratheodory mapping,

$$\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times V\text{-}P_f(X)},$$

there exists a uC-correspondence, $\eta(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times V\text{-}P_f(X)}$, such that

$$\eta(\omega,\cdot) \in [\mathcal{N}(\omega,\cdot)] \text{ for all } \omega.$$

Proof: First, by Kucia and Nowak (2000), the mapping

$$\omega \longrightarrow Gr\mathcal{N}(\omega,\cdot) \in P_f(V\times X)$$

is measurable. Let $P_f(V\times X)$ be the collection of all nonempty, $\rho_{V\times X}$-closed (and hence, $\rho_{V\times X}$-compact) subsets of $V\times X$ and equip $P_f(V\times X)$ with the Hausdorff metric, $h_{\rho_{V\times X}}$ induced by the sum metric $\rho_{V\times X} := \rho_V + \rho_X$ on $V\times X$. The hyperspace, $(P_f(V\times X), h_{\rho_{V\times X}})$ is a compact metric space. Following Holy (2007), equip the space of all minimal USCOs, $\mathcal{M}_{V\text{-}P_f(X)}$, with the sup metric,

$$\rho_{\sup}(\mathcal{N}^0,\mathcal{N}^1) := \sup\{h_{\rho_X}(\mathcal{N}^0(v),\mathcal{N}^1(v)) : v\in V\}. \quad (22)$$

From Holy-Vadoric (2007), we know that for $\mathcal{N}^0$ and $\mathcal{N}^1$ contained in $\mathcal{U}_{V\text{-}P_f(X)}$,

$$h_{\rho_{V\times X}}(Gr\mathcal{N}^0(\cdot), Gr\mathcal{N}^1(\cdot)) \le \rho_{\sup}(\mathcal{N}^0,\mathcal{N}^1). \quad (23)$$

Thus, $\mathcal{N} \longrightarrow Gr\mathcal{N}(\cdot)$ is $\rho_{\sup}$-$h_{\rho_{V\times X}}$-continuous.[8]

Letting $G(V,P_f(X))$ denote the collection of all set-valued mappings with $\rho_{V\times X}$-closed graphs, because (V,ρ_V) and (X,ρ_X) are compact, $(G(V,P_f(X)),\rho_{\sup})$ is compact. We also know via results in Holy (2007) that $(\mathcal{U}_{V-P_f(X)},\rho_{\sup})$ is a closed (and hence compact) set in $(G(V,P_f(X)),\rho_{\sup})$ and moreover, that $(\mathcal{M}_{V-P_f(X)},\rho_{\sup})$ is a closed (and hence compact) set in $(\mathcal{U}_{V-P_f(X)},\rho^*_{\sup})$. Consider the minimization problem

$$\min_{\eta(\cdot)\in\mathcal{M}_{V\text{-}P_f(X)}} e_{\rho_{V\times X}}(Gr\eta(\cdot), Gr\mathcal{N}_\omega(\cdot)). \quad (24)$$

For each ω the function,

$$\eta(\cdot) \longrightarrow e_{\rho_{V\times X}}(Gr\eta(\cdot), Gr\mathcal{N}_\omega(\cdot)), \quad (25)$$

is real-valued and $\rho^*_{\sup}$-continuous on the $\rho^*_{\sup}$-compact metric space, $\mathcal{M}_{V\text{-}P_f(X)}$.[9] Moreover, for each $\eta(\cdot)\in\mathcal{M}_{V\text{-}P_f(X)}$, the function,

$$\omega \longrightarrow e_{\rho_{V\times X}}(Gr\eta(\cdot), Gr\mathcal{N}_\omega(\cdot)), \quad (26)$$

is real-valued and B_Ω-measurable on ω. Thus, by optimal measurable selection (see Wagner, 1977), there exists a measurable selection, $\omega \longrightarrow \eta_\omega(\cdot)\in\mathcal{M}_{V\text{-}P_f(X)}$ such that

$$e_{\rho_{V\times X}}(Gr\eta_\omega(\cdot), Gr\mathcal{N}_\omega(\cdot)) = \min_{\eta(\cdot)\in\mathcal{M}_{V\text{-}P_f(X)}} e_{\rho_{V\times X}}(Gr\eta(\cdot), Gr\mathcal{N}_\omega(\cdot)). \quad (27)$$

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[8] If $\rho_{\sup}(\Gamma^n(\cdot),\Gamma^*(\cdot)) \longrightarrow 0$, then

$$h_{\rho_{V\times X}}(Gr\Gamma^n(\cdot), Gr\Gamma^*(\cdot)) \longrightarrow 0,$$

for $\{\Gamma^n(\cdot)\}_n \subset \mathcal{U}_{V\text{-}X}$ and $\Gamma^*(\cdot)\in\mathcal{U}_{V\text{-}X}$.

[9] If $\rho_{\sup}(\eta^n(\cdot),\eta(\cdot)) \longrightarrow 0$, then for each t

$$h_{\rho_{V\times X}}(Gr\Gamma_t(\cdot), Gr\eta^n(\cdot)) \longrightarrow h_{\rho_{V\times X}}(Gr\Gamma_t(\cdot), Gr\eta(\cdot)).$$

By the properties of the excess function on $P_f(V \times X)$, we have for all (ω, v),

$$\eta_\omega(v) \subset \mathcal{N}_\omega(v) := \mathcal{N}(\omega, v),$$

Thus, we have that $\eta_{(\cdot)}(\cdot)$ is upper Caratheodory, and because for all (ω, v),

$$\eta_\omega(v) \subset \mathcal{N}(\omega, v) \text{ and } \eta_\omega(\cdot) \in \mathcal{M}_{V-P_f(X)},$$

we have for each ω, $\eta_\omega(\cdot) \in [\mathcal{N}(\omega, \cdot)]$ - implying that $\mathcal{MUC}^{\mathcal{N}} \neq \varnothing$. **Q.E.D.**

2.2.4 ε-Caratheodory Selections for Minimal uC Correspondences

We begin with the basic definitions.

Definitions 1 *(Caratheodory Functions)*
A function, $g(\cdot,\cdot) : \Omega \times V \longrightarrow X$, is said to be Caratheodory if (i) for each $\omega \in \Omega$ the function, $g(\omega, \cdot) := g_\omega(\cdot)$, defined on V with values in X is ρ_V-ρ_X-continuous, and if (ii) for each $v \in V$ the function, $g(\cdot, v) := g_v(\cdot)$ is (B_Ω, B_X)-measurable.

Definitions 2 *(ε-Caratheodory Selections of uC Correspondences)*
An upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega \times V\text{-}P_f(X)}$, has an ε-Caratheodory selection if there is a sub-correspondence, $\phi(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$ belonging to $\mathcal{N}(\cdot,\cdot)$ such that for each $\varepsilon > 0$ there is a Caratheodory function, $g^\varepsilon(\cdot,\cdot) : \Omega \times V \longrightarrow X$, having the property that for each $(\omega, v) \in \Omega \times V$ and each $(v, g^\varepsilon(\omega, v)) \in V \times X$ there exists $(\overline{v}, \overline{x}) \in Gr\phi(\omega, \cdot)$ such that

$$\rho_V(v, \overline{v}) + \rho_X(g^\varepsilon(\omega, v), \overline{x}) < \varepsilon,$$

or equivalently, a Caratheodory function, $g^\varepsilon : \Omega \times V \longrightarrow X$, such that for each ω

$$Grg^\varepsilon(\omega, \cdot) \subset B_{\rho_{V \times X}}(\varepsilon, Gr\phi(\omega, \cdot)).$$

We say that the uC correspondence, $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega \times V\text{-}P_f(X)}$, is Caratheodory approximable if $\mathcal{N}(\cdot,\cdot)$ has a uC subcorrespondence, $\phi(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$, such that for each $\varepsilon > 0$, $\phi(\cdot,\cdot)$ has an ε-Caratheodory Selection.

By Theorem 4.2 in Kucia and Nowak (2000), a sufficient condition for $\mathcal{N}(\cdot,\cdot)$ to have for each $\varepsilon > 0$ an ε-Caratheodory selection is for each ω the minimal uC correspondence, $\eta(\omega, \cdot)$, to have an ε-continuous selection.

Definitions 3 *(ε-Continuous Selections of Minimal USCOs, $\eta(\omega, \cdot)$)*
Given $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega \times V\text{-}P_f(X)}$, let $\phi(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$. A function, $f^\varepsilon(\cdot) : V \longrightarrow X$, is said to be ε-continuous selection of $\phi(\omega, \cdot)$ if for each $v \in V$ and each $(v, f^\varepsilon(v)) \in V \times X$ there exists $(\overline{v}, \overline{x}) \in Gr\eta\phi(\omega, \cdot)$ such that

$$\rho_V(v, \overline{v}) + \rho_X(f^\varepsilon(v), \overline{x}) < \varepsilon,$$

or equivalently, such that the continuous function, $f^\varepsilon : V \longrightarrow X$, has the property that

$$Grf^\varepsilon(\cdot) \subset B_{\rho_{V \times X}}(\varepsilon, Gr\phi(\omega, \cdot)).$$

We say that the uC subcorrespondence, $\phi(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$, is approximable if for each ω and for each $\varepsilon > 0$, the sub-USCO, $\phi(\omega, \cdot)$, has an ε-continuous selection.

Our first objective is to identify conditions that the uC Nash correspondence $\mathcal{N}(\cdot,\cdot)$ must satisfy in order to guarantee that some Nash sub-correspondence $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$ is continuum valued. In fact, we will show that under assumptions $[\mathcal{PSG}]$ if $\eta(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{N}}$ has the $3M$ property, then for each ω, $\eta(\omega,\cdot)$ is a continuum valued USCO such that for each v, $\eta(\omega,v)$ is a continuum of essential Nash equilibria. Once we have achieved our first objective, our second objective will be to show that all parameterized, state-contingent games satisfying assumptions $[\mathcal{PSG}]$ have minimal uC Nash correspondences having the $3M$ property, and therefore, continuum valued minimal uC Nash correspondences,

3 $3M$ Minimal Nash Correspondences Are Continuum Valued

A profile of player action choices, $x^* \in \Phi(\omega, v_d)$, is a Nash equilibrium for the (ω,v)-game, $\mathcal{G}_{(\omega,v)}$, if for each player $d \in D$

$$u_d(\omega, v_d, (x_d^*, x_{-d}^*)) = \max_{x_d \in \Phi_d(\omega,v_d)} u_d(\omega, v_d, (x_d, x_{-d}^*)).$$

Under assumptions $[\mathcal{PSG}]$ for each $(\omega,v) \in \Omega \times V$ the (ω,v)-game, $\mathcal{G}_{(\omega,v)}$, has a nonempty, ρ_X-compact set of Nash equilibria, $\mathcal{N}(\omega,v)$, and using Berge's Maximum Theorem it is straightforward to show that the *Nash correspondence*,

$$\mathcal{N}(\cdot,\cdot) : \Omega \times V \longrightarrow P_f(X)$$

is upper Caratheodory, $(B_\Omega \times B_V, B_X)$-measurable in (ω,v) and ρ_V-ρ_X-upper semicontinuous in v with nonempty, ρ_X-compact values. Here, recall that $P_f(X)$ denotes the collection of nonempty, ρ_X-closed (and hence, ρ_X-compact) subsets of X.

3.1 Essential Sets, the $3M$ Property, and Connected Values

We begin with the definitions of essential and minimally essential values belonging to an upper Caratheodory correspondence's USCO part for a particular ω - where essential and minimally essential are in the sense of Fort (1950). Then we introduce the $3M$ property and we show that, given the primitives and assumptions above (call these $\mathcal{PSG}$), all minimal uC correspondences have USCO parts that are minimally essentially valued and if they have the $3M$ property, they are continuum valued and if they are continuum valued, they are $3M$. Later, when we introduce an extended notion of Ky Fan sets and our decomposition of the upper Caratheodory Nash correspondence of a parameterized, state-contingent game, we will revisit the $3M$ property - in the hyperspace of Ky Fan sets. More importantly, we will show, given the primitives and assumptions made above, that all parameterized, state-contingent games have $3M$ minimal uC Nash sub-correspondences.

3.1.1 Essential Sets and the $3M$ Property

Let $\mathcal{N}(\cdot,\cdot)$ be an upper Caratheodory correspondence with USCO part,

$$\mathcal{N}^{USCO} := \left\{ \mathcal{N}_\omega(\cdot) \in \mathcal{U}_{V-P_f(X)} : \omega \in \Omega \right\}.$$

Definitions 4 *(Essential Sets)*
(1) (Essential Set) A nonempty closed subset $e_\omega(v^0)$ of $\mathcal{N}(\omega,v^0)$ is said to be essential for $\mathcal{N}_\omega(\cdot)$ at $v^0 \in V$ if for each $\varepsilon > 0$ there exists $\delta^\varepsilon > 0$ such that for all $v \in B_{\rho_V}(\delta^\varepsilon, v^0)$,

$$\mathcal{N}(\omega,v) \cap B_{\rho_X}(\varepsilon, e_\omega(v^0)) \neq \emptyset. \tag{28}$$

We will denote by $\mathcal{E}[\mathcal{N}(\omega, v^0)] \subset P_f(\mathcal{N}(\omega, v^0))$ the collection of all nonempty, ρ_X-closed subsets of $\mathcal{N}(\omega, v^0)$ essential for $\mathcal{N}_\omega(\cdot)$ at $v^0 \in V$.
(2) (Minimal Essential Set) A nonempty closed subset $m_\omega(v^0)$ of $\mathcal{N}(\omega, v^0)$ is said to be minimally essential for $\mathcal{N}_\omega(\cdot)$ at $v^0 \in V$ if (i) $m_\omega(v^0) \in \mathcal{E}[\mathcal{N}(\omega, v^0)]$ and if (ii) $m_\omega(v^0)$ is a minimal element of $\mathcal{E}[\mathcal{N}(\omega, v^0)]$ ordered by set inclusion (i.e., if $e_\omega(v^0) \in \mathcal{E}[\mathcal{N}(\omega, v^0)]$ and $e_\omega(v^0) \subseteq m_\omega(v^0)$ then $e_\omega(v^0) = m_\omega(v^0)$). We will denote by $\mathcal{E}^*[\mathcal{N}(\omega, v^0)]$ the collection of all nonempty, closed subsets of $\mathcal{N}(\omega, v^0)$ minimally essential for $\mathcal{N}_\omega(\cdot)$ at $v^0 \in V$.

The $3M$ property (i.e., the 3 misses property) is defined as follows:

Definition 5 *(The $3M$ Property)*
Let $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times V - P_f(X)}$ be an upper Caratheodory correspondence and consider the minimal upper Caratheodory correspondence, $\eta(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{N}}$. We say that $\eta(\cdot,\cdot)$ is $3M$ at (ω^0, v^0) if the minimal ω^0-USCO, η_{ω^0}, has the property at v^0 that for any $\delta > 0$ and for any pair of nonempty disjoint closed sets, F^1 and F^2, in X there exists v^1 and v^2 in $B_{\rho_V}(\delta, v^0)$ such that

$$\eta_{\omega^0}(v^1) \cap F^1 = \emptyset \text{ and } \eta_{\omega^0}(v^2) \cap F^2 = \emptyset, \tag{29}$$

then there exists a third point, v^3, in the larger open ball, $B_{\rho_V}(3\delta, v^0)$, such that

$$\eta_{\omega^0}(v^3) \cap [F^1 \cup F^2] = \emptyset. \tag{30}$$

We say that an upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, is $3M$ if for some $\eta(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{N}}$, $\eta(\cdot,\cdot)$ is $3M$ at (ω, v) for all $(\omega, v) \in \Omega \times V$. We will denote by $\mathcal{MUC}^{\mathcal{N}}_{3M}$ the collection of all $3M$ minimal uC correspondences belonging to $\mathcal{N}(\cdot,\cdot)$, and we will denote by $\mathcal{UC}^{3M}_{\Omega\times V\text{-}P_f(X)}$, the collection of all uC correspondences such that $\mathcal{MUC}^{\mathcal{N}}_{3M} \neq \varnothing$.

We say that an upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, is quasi-minimal at ω, if the ω-USCO, $\mathcal{N}_\omega(\cdot) := \mathcal{N}(\omega, \cdot)$ is quasi-minimal. We say that $\mathcal{N}(\cdot,\cdot)$ is quasi-minimal if the ω-USCO, $\mathcal{N}_\omega(\cdot) := \mathcal{N}(\omega, \cdot)$ is quasi-minimal for all ω. We will denote the collection of quasi-minimal upper Caratheodory correspondences by $\mathcal{Q}_{\Omega\times V\text{-}P_f(X)}$. The following Theorem establishes a fundamental fact about quasi-minimal upper Caratheodory correspondences: any minimal uC correspondence belonging to a quasi-minimal uC correspondence takes minimally essential values.

Theorem 2 *(The Quasi-Minimal Theorem for uC Correspondences)*
Suppose assumptions [$\mathcal{PSG}$] hold. Let $\mathcal{N}(\cdot,\cdot)$ be a quasi-minimal upper Caratheodory correspondence. Then, for $\eta(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{N}}$, $\eta(\omega, v) \in \mathcal{E}^*[\mathcal{N}(\omega, v)]$ for each $(\omega, v) \in \Omega \times V$.

Proof: Suppose that for some $(\omega^0, v^0) \in \Omega \times V$ there is some nonempty, closed and proper subset $e_{\omega^0}(v^0)$ of $\eta(\omega^0, v^0)$ with $e_{\omega^0}(v^0) \in \mathcal{E}[\mathcal{N}(\omega^0, v^0)]$. Fix some $x^0 \in \eta(\omega^0, v^0) \setminus e_{\omega^0}(v^0)$ and let $B_{\rho_X}(\varepsilon^0, e_{\omega^0}(v^0))$ be an open enlargement of $e_{\omega^0}(v^0)$ such that $x^0 \notin \overline{B}_{\rho_X}(\varepsilon^0, e_{\omega^0}(v^0))$. Since $e_{\omega^0}(v^0) \in \mathcal{E}[\mathcal{N}(\omega^0, v^0)]$ there is a $\delta^0 > 0$ such that for all $v \in B_{\rho_V}(\delta^0, v^0)$, $\mathcal{N}_{\omega^0}(v) \cap B_{\rho_X}(\varepsilon^0, e_{\omega^0}(v^0)) \neq \emptyset$. Define the mapping $\varphi_{\omega^0}(\cdot)$ as follows:

$$\varphi_{\omega^0}(v) := \begin{cases} \mathcal{N}_{\omega^0}(v) \cap \overline{B}_{\rho_X}(\varepsilon^0, e_{\omega^0}(v^0)) & v \in B_{\rho_V}(\delta^0, v^0) \\ \mathcal{N}_{\omega^0}(v) & v \in V \setminus B_{\rho_V}(\delta^0, v^0). \end{cases} \tag{31}$$

By Lemma 2(ii) in Anguelov and Kalenda (2009), $\varphi_{\omega^0}(\cdot)$ is an USCO with $Gr\varphi_{\omega^0} \subset Gr\mathcal{N}_{\omega^0}$ and hence $Gr\eta_{\omega^0} \subset Gr\varphi_{\omega^0}$. In particular, $x^0 \in \varphi_{\omega^0}(v^0)$, a contradiction. **Q.E.D.**

If $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times V\text{-}P_f(X)}$ is quasi-minimal, then it can contain only one sub uC correspondence, namely, its minimal uC correspondence.

Theorem 3 *(Quasi-minimal USCOs and minimal USCOs)*
Suppose assumptions [$\mathcal{PSG}$] hold. Let $\mathcal{N}(\cdot,\cdot)$ be a quasi-minimal upper Caratheodory correspondence. If $\eta(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{N}}$, then for any $\phi \in \mathcal{U}_{\rho_V\text{-}\rho_X}$ with $Gr\phi$ is a proper subset of $Gr\mathcal{N}_{\omega^0}$ for some ω^0, $Gr\phi = Gr\eta_{\omega^0}$.

Proof: First, suppose that $(v^0, x^0) \in Gr\phi$, but $(v^0, x^0) \notin Gr\eta_{\omega^0}$. Thus, we have $x^0 \notin \eta_{\omega^0}(v^0)$. Because $\eta_{\omega^0}(v^0)$ is ρ_X-closed, there is a closed ball, $\overline{B}_{\rho_X}(\varepsilon_{x^0}, x^0)$ of sufficiently small radius $\varepsilon_{x^0} > 0$, such that $\overline{B}_{\rho_X}(\varepsilon_{x^0}, x^0) \cap \eta_{\omega^0}(v^0) = \emptyset$.

Consider the correspondence, $\phi^0_\delta : V \longrightarrow X$, given by

$$\phi^0_\delta(v) := \begin{cases} \phi(v) \cap \overline{B}_{\rho_X}(\varepsilon_{x^0}, x^0) & v \in B_{\rho_V}(\delta, v^0) \\ \phi(v) & v \in V \backslash B_{\rho_V}(\delta, v^0). \end{cases} \tag{32}$$

By Lemma 2(ii) in Anguelov and Kalenda (2009), $\phi^0_\delta(\cdot)$ is an USCO provided $\phi^0_\delta(v) \neq \emptyset$ for all $v \in V$. To show that this is true, it suffices to show that for some $\delta^0 > 0$,

$$\phi(v) \cap \overline{B}_{\rho_X}(\varepsilon_{x^0}, x^0) \neq \emptyset \text{ for all } v \in B_{\rho_V}(\delta, v^0). \tag{33}$$

Suppose that (33) is false. Thus for each n, there exists $v^n \in B_{\rho_V}(\frac{1}{n}, v^0)$ such that

$$dist_{\rho_X}(x^0, \phi(v^n)) := \min_{x \in \phi(v^n)} \rho_X(x^0, x) > \varepsilon_{x^0}.$$

For each n, the closest point $x \in \phi(v^n)$ to x^0 is at a ρ_X-distance from x^0 greater than ε_{x^0}. Thus, no point in $\phi(v^n)$ is contained in the closed ball $\overline{B}_{\rho_X}(\varepsilon_{x^0}, x^0)$. Thus, for v^n not equal to v^0 but arbitrarily close to v^0, $dist_{\rho_X}(x^0, \phi(v^n)) > \varepsilon_{x^0}$, but at v^0, $dist_{\rho_X}(x^0, \phi(v^0)) = 0$. We will show that this jump discontinuity leads to a contradiction. First note that because the function $\rho_X(x^0, \cdot)$ is ρ_X-continuous on X, for each $x' \in \phi(v^0)$ there exists $\varepsilon_{x'} > 0$ and an ρ_X-open ball, $B_{\rho_X}(\varepsilon_{x'}, x')$ such that for all $x \in B_{\rho_X}(\varepsilon_{x'}, x')$,

$$\rho_X(x^0, x) \leq \rho_X(x^0, x') + \varepsilon_{x^0}.$$

Thus, we have $\phi(v^0) \subset \cup_{x' \in \phi(v^0)} B_{\rho_X}(\varepsilon_{x'}, x')$ implying via the ρ_X-compactness of $\phi(v^0)$ that there are finitely many balls,

$$\left\{B_{\rho_X}(\varepsilon_{x^0}, x^0), B_{\rho_X}(\varepsilon_{x^1}, x^1), \ldots, B_{\rho_X}(\varepsilon_{x^n}, x^n)\right\},$$

covering $\phi(v^0)$, where $\{x^0, x^1, \ldots, x^n\} \subset \phi(v^0)$. Given that $\phi(\cdot)$ is USCO, there exists $\delta_{v^0} > 0$ such that for all $v \in B_{\rho_V}(\delta_{v^0}, v^0)$,

$$\phi(v) \subset \cup_{i=0}^n B_{\rho_X}(\varepsilon_{x^i}, x^i).$$

Thus, if $v \in B_{\rho_V}(\delta_{v^0}, v^0)$ and $x \in \phi(v)$, then $x \in B_{\rho_X}(\varepsilon_{x^i}, x^i)$ for some $i = 0, 1, \ldots, n$, and therefore, we have for all $x \in B_{\rho_X}(\varepsilon_{x^i}, x^i)$

$$dist_{\rho_X}(x^0, \phi(v)) := \min_{x \in \phi(v)} \rho_X(x^0, x) \leq \rho_X(x^0, x) \leq \rho_X(x^0, x^i) + \varepsilon_{x^0}.$$

Because

$$dist_{\rho_X}(x^0, \phi(v^0)) := \min_{x \in \phi(v^0)} \rho_X(x^0, x) = 0,$$

we have for $v \in B_{\rho_V}(\delta_{v^0}, v^0)$,

$$dist_{\rho_X}(x^0, \phi(v)) \leq \min_{0 \leq i \leq n} \rho_X(x^0, x^i) + \varepsilon_{x^0} \leq dist_{\rho_X}(x^0, \phi(v^0)) + \varepsilon_{x^0} = \varepsilon_{x^0}.$$

Thus, we have a contradiction and we must conclude that (33) is true for $\delta_{x^0} > 0$. Therefore, $\phi^0_{\delta_{v^0}}(\cdot) \in \mathcal{U}_{\rho_V\text{-}\rho_X}$. Letting φ be any minimal USCO contained in $[\phi^0_{\delta_{v^0}}]$, we have a contradiction: $[\mathcal{N}_{\omega^0}(\cdot)]$ contains at least two different minimal USCO maps, η_{ω^0} and φ - but $[\mathcal{N}_{\omega^0}(\cdot)] = \{\eta_{\omega^0}(\cdot)\}$. Therefore, we must conclude that $Gr\phi = Gr\eta_{\omega^0}$.**Q.E.D.**

Let $\mathcal{MUC}^{\mathcal{N}}_{3M}$ denote the collection of minimal uCs, $\eta(\cdot,\cdot)$, belonging to $\mathcal{N} \in \mathcal{UC}_{\Omega\times V-P_f(X)}$ such that for each $\omega \in \Omega$, $\eta(\omega,\cdot)$, an USCO defined on V taking nonempty closed values in X, is $3M$. Let $\mathcal{MUC}^{\mathcal{N}}_{C(X)}$ denote the collection of minimal uCs, $\eta(\cdot,\cdot)$, belonging to $\mathcal{N} \in \mathcal{UC}_{\Omega\times V\text{-}P_f(X)}$ such that for each $\omega \in \omega$, $\eta(\omega,\cdot)$, an USCO defined on V taking nonempty *closed and connected* values in X - i.e., values in $C_f(X)$, where $C_f(X)$ denotes the hyperspace of nonempty closed and connected subsets of X (i.e., subcontinua of X). Our next Theorem establishes that for any minimal uC correspondence, $\eta(\cdot,\cdot)$, defined on an M-convex, compact metric space taking nonempty, ρ_X-closed values in X,

$$\eta(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{N}}_{3M} \text{ if and only if } \eta(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{N}}_{C(X)}.$$

Theorem 4 *(For $\mathcal{N} \in \mathcal{UC}_{\Omega\times V-P_f(X)}$, $\mathcal{MUC}^{\mathcal{N}}_{3M} = \mathcal{MUC}^{\mathcal{N}}_{C(X)}$)*
Let (Ω, B_Ω) be a measurable space of states with Borel σ-field B_Ω and let (V,ρ_V) and (X,ρ_X) be Peano continua with M-convex metrics ρ_V and ρ_X. Given any $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times V\text{-}P_f(X)}$, any $\eta(\cdot,\cdot) \in \mathcal{MUC}^{\mathcal{N}}$, and any state-parameter pair, $(\omega, v) \in \Omega \times V$, the following statements are equivalent:

$$\left.\begin{array}{c} (1)\ \eta_\omega(\cdot) \text{ is } 3M \text{ at } v. \\ \\ (2)\ \eta_\omega(v) \text{ is connected.} \end{array}\right\} \tag{34}$$

Before we present the proof, note that if $\eta_{\omega^0}(\cdot) \in [\mathcal{N}_{\omega^0}(\cdot)]$, then $\eta_{\omega^0}(\cdot)$ is quasi-minimal relative to $\eta_{\omega^0}(\cdot)$. Thus, by the Quasi-Minimal Theorem 2 above, we have for $\eta_{\omega^0}(\cdot) \in [\mathcal{N}_{\omega^0}(\cdot)]$ that $\eta_{\omega^0}(\cdot) \in \mathcal{E}[\mathcal{N}_{\omega^0}(\cdot)]$ with $\eta_{\omega^0}(\cdot) \in \mathcal{E}^*[\eta_{\omega^0}(\cdot)]$.

Proof: (1) $\Longrightarrow$ (2). We will show that if for some (ω^0, v^0), $\eta_{\omega^0}(v^0)$ is not connected, then η is not $3M$ at (ω^0, v^0). In particular, we will show that there is an open ball, $B_{\rho_V}(\delta, v^0)$ and two disjoint closed sets, F^1 and F^2 such that for $v^{\delta 1}$ and $v^{\delta 2}$ in $B_{\rho_V}(\delta, v^0)$,

$$\eta_{\omega^0}(v^{\delta 1}) \cap F^1 = \varnothing \text{ and } \eta_{\omega^0}(v^{\delta 2}) \cap F^2 = \varnothing,$$

but such that $\eta(B_{\rho_V}(\delta, v^0)) \subset F^1 \cup F^2$ - implying that η is not $3M$ at (ω^0, v^0).

To this end, suppose that for some (ω^0, v^0), $\eta_{\omega^0}(v^0)$ is not connected. Let $\eta_{\omega^0}(v^0) = \eta^1_{\omega^0}(v^0) \cup \eta^2_{\omega^0}(v^0)$ for two nonempty, disjoint closed sets, $\eta^1_{\omega^0}(v^0)$ and $\eta^2_{\omega^0}(v^0)$. We can find two nonempty, *disjoint* open subsets, W^1 and W^2, in X such that (i) $\eta^1_{\omega^0}(v^0) \subset W^1$ and $\eta^2_{\omega^0}(v^0) \subset W^2$, and (ii) $\overline{W}^1 \cap \overline{W}^2 = \varnothing$. By the upper semicontinuity of $\eta_{\omega^0}(\cdot)$, we have for some $\delta' > 0$,

$$\eta_{\omega^0}(B_{\rho_V}(\delta', v^0)) \subset W^1 \cup W^2.$$

Because $\eta(\cdot,\cdot)$ is quasi-minimal relative to $\eta(\cdot,\cdot)$, we have by the Quasi-Minimal Theorem above that *for each* $(\omega, v) \in \Omega \times V$, $\eta(\omega, v) \in \mathcal{E}^*[\mathcal{N}(\omega, v)]$ implying that neither $\eta^1_{\omega^0}(v^0)$ nor $\eta^2_{\omega^0}(v^0)$ are essential for $\eta_{\omega^0}(\cdot)$ at v^0. Thus, there are two nonempty, disjoint open sets G^1 and G^2 with

$$\eta^1_{\omega^0}(v^0) \subset G^1 \text{and } \eta^2_{\omega^0}(v^0) \subset G^2$$

such that *for all* $\delta > 0$, *there exists* $v^{\delta 1}$ and $v^{\delta 2}$ in $B_{\rho_V}(\delta, v^0)$ such that

$$n_{\omega^0}(v^{\delta 1}) \cap G^1 = \emptyset \text{ and } n_{\omega^0}(v^{\delta 2}) \cap G^2 = \emptyset.$$

Let $U^1 = W^1 \cap G^1$ and $U^2 = W^2 \cap G^2$. We have $U^1$and U^2 disjoint open sets such that $\eta^1_{\omega^0}(v^0) \subset U^1$ and $\eta^2_{\omega^0}(v^0) \subset U^2$ and for all $\delta > 0$, there exist

$$v^{\delta 1} \in B_{\rho_V}(\delta, v^0) \text{ and } v^{\delta 2} \in B_{\rho_V}(\delta, v^0)$$

such that

$$\eta_{\omega^0}(v^{\delta 1}) \cap U^1 = \emptyset \text{ and } \eta_{\omega^0}(v^{\delta 2}) \cap U^2 = \emptyset. \quad (35)$$

Given that the sets $\eta_{\omega^0}(v^{\delta i})$ are compact, there exists open sets V^1 and V^2 such that for $i = 1, 2$,

$$\eta^i_{\omega^0}(v^0) \subset V^i \subset \overline{V}^i \subset U^i.$$

Thus, we have *for all* $\delta > 0$, $v^{\delta i} \in B_{\rho_V}(\delta, v^0)$ such that

$$\eta_{\omega^0}(v^{\delta 1}) \cap \overline{V}^1 = \emptyset \text{ and } \eta_{\omega^0}(v^{\delta 2}) \cap \overline{V}^2 = \emptyset. \quad (36)$$

Because $\eta_{\omega^0}(v^0)$ is minimally essential for $\eta_{\omega^0}(\cdot)$ at v^0 (see Theorem 2 above) and because $\eta_{\omega^0}(v^0) \subset \left[V^1 \cup V^2\right]$, there exists a $\delta^* > 0$ such that *for all* $v \in B_{\rho_V}(\delta^*, v^0)$,

$$\eta_{\omega^0}(v) \cap \left[V^1 \cup V^2\right] \neq \emptyset, \quad (37)$$

implying that *for all* $v \in B_{\rho_V}(\delta^*, v^0)$,

$$\eta_{\omega^0}(\overline{v}) \cap \left[\overline{V}^1 \cup \overline{V}^2\right] \neq \emptyset. \quad (38)$$

But because $\delta > 0$ can be chosen arbitrarily, choosing $\delta = \frac{\delta^*}{3}$, we have

$$\eta_{\omega^0}(v^{\frac{\delta^*}{3} 1}) \cap \overline{V}^1 = \emptyset \text{ and } \eta_{\omega^0}(v^{\frac{\delta^*}{3} 2}) \cap \overline{V}^2 = \emptyset,$$

But for no

$$\overline{v} \in B_{\rho_V}(3\tfrac{\delta^*}{3}, v^0) = B_{\rho_V}(\delta^*, v^0),$$

is it true that

$$\eta_{\omega^0}(\overline{v}) \cap \left[\overline{V}^1 \cup \overline{V}^2\right] = \emptyset. \quad (39)$$

Thus, η is not $3M$ at (ω^0, v^0).

(2) $\Longrightarrow$ (1). We will show directly that if $\eta_{\omega^0}(v^0)$ is connected, then η *is* $3M$ at (ω^0, v^0). Suppose that $\eta_{\omega^0}(v^0)$ is connected. Let $B_{\rho_V}(\delta^1, v^0)$ be an open ball about v^0, $\delta^1 > 0$, and let F^1 and F^2 be any two disjoint closed sets in X.

First, suppose that $\eta_{\omega^0}(v^0) \backslash [F^1 \cup F^2] \neq \emptyset$. By the minimality of η_{ω^0} and part (3) of the *Characterizations of Minimal USCOs* above for some open subset G of $B_{\rho_V}(\delta^1, v^0)$,

$$\eta_{\omega^0}(G) \cap [F^1 \cup F^2] = \varnothing.$$

Taking $\delta^0 = \delta^1$, we see that η is $3M$ at (ω^0, v^0).

Second, suppose that $\eta_{\omega^0}(v^0) \subset [F^1 \cup F^2]$. Since $\eta_{\omega^0}(v^0)$ is connected we can assume that $\eta_{\omega^0}(v^0) \subset F^1$. By the upper semicontinuity of $\eta_{\omega^0}(\cdot)$ there is $\delta^0 < \delta^1$ such that $\eta_{\omega^0}(B_{\rho_V}(\delta^0, v^0)) \cap F^2 = \varnothing$. Again, we see that η is $3M$ at (ω^0, v^0). **Q.E.D.**

4 All Parameterized State-Contingent Games Have the $3M$ Property

Our main objective now is to show that all parametrized state-contingent games satisfying assumptions $[\mathcal{PSG}]$ introduced above have upper Caratheodory Nash correspondences having the $3M$ property. A key ingredient in our approach is our novel decomposition of the Nash correspondence. In particular, we show that all $\mathcal{PSG}s$ satisfying assumptions

[$\mathcal{PSG}$] have Nash correspondences that can be written as the composition of the *Ky Fan correspondence*,

$$N(\cdot): Ky\ Fan\ Sets \longrightarrow Sets\ of\ Nash\ Equilibria,$$

with the collective security function,

$$K(\cdot,\cdot): \textit{ State-Parameter Pairs} \longrightarrow Ky\ Fan\ Sets$$

In particular, we show that for any parameterized, state-contingent game satisfying [$\mathcal{PSG}$], the upper Caratheodory Nash correspondences, $\mathcal{N}(\cdot,\cdot)$, is given by the composition mapping,

$$(\omega, v) \longrightarrow \mathcal{N}(\omega, v) = N \circ K(\omega, v),$$

where $N(\cdot)$ is an USCO mapping from Ky Fan sets into Nash equilibrium action profiles and $K(\cdot,\cdot)$ is a Caratheodory function from state-parameter pairs into Ky Fan sets (B_Ω-$B_\mathbb{S}$-measurable in ω and ρ_V-$h_\mathbb{S}$-continuous in v). While the collective security function, $K(\cdot,\cdot)$, is game specific, for each (ω, v), encoding in the game's Ky Fan set the relevant details of the game, *the KFC, $N(\cdot)$, is universal and common to all strategic form games.*

4.1 Ky Fan Sets and the Ky Fan Correspondence

We begin by introducing an expanded notion of Ky Fan sets, and then present several results concerning the properties of the hyperspace of Ky Fan sets and the Ky Fan Correspondence - i.e., the KFC. The big surprise here is that all $KFCs$, $N(\cdot): \mathbb{S} \longrightarrow P_f(X)$, from the compact metric hyperspace, $\mathbb{S}$, of Ky Fan sets into nonempty, closed sets of Nash equilibria are $3M$ USCOs possessing minimal USCOs (i.e., minimal $KFCs$), $n(\cdot)$, *all of which are* $3M$. Hence, by Theorem 4 above, all minimal $KFCs$ are continuum-valued and when composed with the collective security mapping, $K(\cdot,\cdot)$, induce in each state ω a closed, connected-valued minimal USCO, $n(K(\omega,\cdot))$, belonging to the ω-Nash USCO, $\mathcal{N}(\omega,\cdot)$. Moreover, we will show that the composition of a minimal KFC, $n(\cdot) \in [N(\cdot)]$, with the $\mathcal{PSG}'s$ collective security mapping, $K(\cdot,\cdot)$, induces an upper Caratheodory Nash correspondence,

$$(\omega, v) \longrightarrow n(K(\omega, v)),$$

with the property that for all ω,

$$n(K(\omega,\cdot)) \in [\mathcal{N}(\omega,\cdot)].$$

By Theorem 4 above and the results we will establish in this section - and in particular our results establishing that all minimal KFCs, $n(\cdot)$, are $3M$ - we will be able to conclude that $n(K(\cdot,\cdot)) \in \mathcal{MUC}^{\mathcal{N}} \subset \mathcal{UC}^{3M}_{\Omega\times V\text{-}P_f(X)}$ - i.e., that $n(K(\cdot,\cdot))$ is a minimal Nash correspondence belonging to the Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, and that $n(K(\cdot,\cdot))$ has the $3M$ property.

We begin by introducing an expanded notion of Ky Fan sets and a detailed analysis of KFCs.

4.2 The Domain and Range of a Ky Fan Set

Let $P_f\ (X \times X)$ denote the collection of all nonempty, $\rho_{X\times X}$-closed subsets of $X \times X$ and equip $P_f\ (X \times X)$ with the Hausdorff metric, $h_{X\times X}$, induced by the sum metric $\rho_{X\times X} := \rho_X + \rho_X$ on $X \times X$.[10] Given any set $E \in P_f(X \times X)$, define the *domain* of E

[10] Because the metric ρ_X on X is M-convex, the sum metric, $\rho_{X\times X}$, on $X \times X$ is M-convex, implying that the hyperspace $(P_f(X \times X), h_{X\times X})$ is an M-convex continuum (Theorem 4.1 in Duda, 1970).

to be the set

$$\mathcal{D}(E) := \{y \in X : (y,x) \in E \text{ for some } x \in X\} \in P_f(X)$$

Define the *range* of E to be the set

$$\mathcal{R}(E) := \{x \in X : (y,x) \in E \text{ for some } y \in X\} \in P_f(X).$$

The mappings $\mathcal{D}(\cdot)$ and $\mathcal{R}(\cdot)$ are $h_{X\times X}$-h_X-continuous. To see this, simply note that if $h_{X\times X}(E^n, E^0) \longrightarrow 0$, then for every $(y^0, x^0) \in E^0$ there exists a sequence $\{(y^n, x^n)\}_n$ such that $(y^n, x^n) \underset{\rho_{X\times X}}{\longrightarrow} (y^0, x^0)$ and $(y^n, x^n) \in E^n$.

Because the games we will consider are state and parameter contingent, and in particular, because each player in making an optimal choice faces a convex-valued Caratheodory constraint correspondence, $(\omega, v) \longrightarrow \Phi_d(\omega, v_d)$, we must modify the classical definition of a Ky Fan set to take this into account (e.g., see Zhou, Xiang, and Yang, 2005, and Yu, Yang, and Xiang, 2005, for the classical definition).

Definition 6 *(Ky Fan Sets)*
A set $E \in P_f(X \times X)$ is a Ky Fan set if E satisfies the following properties:
(Z1) $\mathcal{D}(E)$ is nonempty, closed, and convex and $\mathcal{D}(E) = \mathcal{R}(E)$;
(Z2) for all $y \in \mathcal{D}(E)$, $(y, y) \in E$;
(Z3) for all $x \in \mathcal{R}(E)$, $\{y \in \mathcal{D}(E) : (y, x) \notin E\}$ is convex (possibly empty).

We will denote by $\mathbb{S}$ the collection of all Ky Fan sets in $P_f(X \times X)$. Thus,

$$\mathbb{S} := \{E \in P_f(X \times X) : \ E \text{ satisfies (Z1)-(Z3)}\}.$$

4.3 The Hyperspace of Ky Fan Sets

Theorem 5 *(The Hyperspace of Ky Fan Sets)*
Suppose assumptions [$\mathcal{PSG}$] hold. Then $\mathbb{S}$ is a $h_{X\times X}$-closed subspace of $P_f(X \times X)$.

Proof: Let $\{E^n\}_n \subset \mathbb{S}$ be a sequence of Ky Fan sets such that $h_{X\times X}(E^n, E^0) \longrightarrow 0$. We must show that $E^0 \in \mathbb{S}$. Because $h_X(\mathcal{D}(E^n), \mathcal{D}(E^0)) \longrightarrow 0$ and because each $\mathcal{D}(E^n) = \mathcal{R}(E^n)$ is convex, we have that $\mathcal{D}(E^0) \in P_{fc}(X)$ and $\mathcal{D}(E^0) = \mathcal{R}(E^0)$.[11] Thus, E^0 satisfies (Z1). Also, note that E^0 satisfies (Z2). The proof will be complete if we can show that for all $x \in \mathcal{R}(E)$, $\{y \in \mathcal{D}(E) : (y, x) \notin E\}$ is convex and possibly empty (i.e., that E^0 satisfies (Z3)). Suppose not. Then for some x^0 in $\mathcal{R}(E^0)$, there exists y^1 and y^2 in $\mathcal{D}(E^0)$, such that for some $y^0 = \lambda^0 y^1 + (1 - \lambda^0) y^2 \in \mathcal{D}(E^0)$, $\lambda^0 \in (0, 1)$, x^0 *deters* y^0 *but does not deter* y^1 or y^2. Therefore, we have $x^0 \notin E^0(y^1) \Longrightarrow (y^1, x^0) \notin E^0$, $x^0 \notin E^0(y^2) \Longrightarrow (y^2, x^0) \notin E^0$, but $x^0 \in E^0(y^0) \Longrightarrow (y^0, x^0) \in E^0$. But now because $h_{X\times X}(E^n, E^0) \longrightarrow 0$, and because $(y^i, x^0) \notin E^0$, for $\delta > 0$ sufficiently small, we have for some N_δ sufficiently large that for any y' on the line segment between y^1 and y^2 contained in each of the convex sets, $\{y \in \mathcal{R}(E^n) : (y, x^0) \notin E^n\}$,

$$[B_{\rho_X}(\delta, y') \times \{x^0\}] \cap E^n = \varnothing \text{ for all } n \geq N_\delta,$$

contradicting the assumption that for $y' = y^0$, $(y^0, x^0) \in E^0$. **Q.E.D.**

[11] $P_{fc}(X)$ denotes the collection of all nonempty, ρ_X-closed and (classically) convex subsets of X.

4.4 $\mathcal{D}$-Equivalence Classes of Ky Fan Sets

Given Ky Fan set $E \in \mathbb{S}$, we define the $\mathcal{D}$-equivalence class, $\mathbb{S}_E$, of Ky Fan sets as follows:

$$\mathbb{S}_E := \{E' \in \mathbb{S} : \mathcal{D}(E') = \mathcal{D}(E)\}. \tag{40}$$

Because $\mathcal{D}(\cdot)$ is $h_{X\times X}$-h_X-continuous, it is easy to show that $\mathbb{S}_E$ is a $h_{X\times X}$-closed subset of $\mathbb{S}$. Thus, if $\{E^n\}_n$ is a sequence of Ky Fan sets in $\mathbb{S}_{\widetilde{E}}$ for some $\widetilde{E} \in \mathbb{S}$, then $E^n \underset{h_{X\times X}}{\longrightarrow} E^0$ implies that $E^0 \in \mathbb{S}_{\widetilde{E}}$. Also, viewing $\mathbb{S}_{(\cdot)}$ as a mapping from $\mathbb{S}$ into $\mathcal{D}$-equivalence classes of Ky Fan sets, it is easy to show that $\mathbb{S}_{(\cdot)}$ has a $h_{X\times X}$-$h_{X\times X}$-closed graph. In particular, if

$$h_{X\times X}(E^n, E^0) + h_{X\times X}(C^n, C^0) \longrightarrow 0,$$

where for all n, $C^n \in \mathbb{S}_{E^n}$, then $C^0 \in \mathbb{S}_{E^0}$. This too is an immediate consequence of the $h_{X\times X}$-h_X-continuity of $\mathcal{D}(\cdot)$.

Let $\mathbb{S}_{(\omega,v)}$ denote the space of Ky Fan sets $\mathcal{D}$-equivalent to the Ky Fan set, $\Phi(\omega,v) \times \Phi(\omega,v)$.[12] For each ω, this $\mathcal{D}$-equivalence class is given by

$$\mathbb{S}_{(\omega,v)} := \mathbb{S}_{\Phi(\omega,v)\times\Phi(\omega,v)} := \{E \in \mathbb{S} : \mathcal{D}(E) = \Phi(\omega,v)\} \subset \mathbb{S}. \tag{41}$$

Because $\Phi(\omega,\cdot) := \Phi_1(\omega,\cdot) \times \cdots \times \Phi_m(\omega,\cdot)$ is a continuous correspondence, it is easy to show that $v \longrightarrow \mathbb{S}_{(\omega,v)}$ has a ρ_V-$h_{X\times X}$-closed graph in $V \times \mathbb{S}$ for each ω, and hence, that $v \longrightarrow \mathbb{S}_{(\omega,v)}$ is upper semicontinuous.

4.5 Ky Fan Correspondences

Consider the correspondence,

$$E \longrightarrow N(E) := \cap_{y\in\mathcal{D}(E)} \{x \in \mathcal{R}(E) : (y,x) \in E\}, \tag{42}$$

defined on $\mathbb{S}$ taking values in $P_f(X)$. We will call the correspondence $N(\cdot)$ from $\mathbb{S}$ into $P_f(X)$ the *Ky Fan Correspondence* (i.e., the KFC). It follows from Lemma 4 in Ky Fan (1961) that if $E \in \mathbb{S}$, then

$$N(E) := \cap_{y\in\mathcal{D}(E)} \{x \in \mathcal{R}(E) : (y,x) \in E\} \neq \emptyset.$$

Theorem 6 *(The KFC is an USCO)*
Under assumptions [$\mathcal{PSG}$], the KFC, $N(\cdot)$ is an USCO, that is,

$$N(\cdot) \in \mathcal{U}_{\mathbb{S}\text{-}X} := \mathcal{U}(\mathbb{S}, P_f(X)). \tag{43}$$

Proof: By Ky Fan (1961) $N(E)$ is nonempty for all $E \in \mathbb{S}$ and it is easy to see that $N(E)$ is compact for all $E \in \mathbb{S}$. To see that $N(\cdot)$ is upper semicontinuous consider a sequence $\{(E^n, x^n)\}_n \subset GrN(\cdot)$ where $\{E^n\}_n \subset \mathbb{S}$ and WLOG assume that $E^n \underset{h_{X\times X}}{\longrightarrow} E^0$, and $x^n \underset{\rho_X}{\longrightarrow} x^0$. By (42) we have for each n, $(y, x^n) \in E^n$ for any $y \in \mathcal{D}(E^n)$. By

[12] Note that for the "box" Ky Fan set

$$\Phi(\omega,v) \times \Phi(\omega,v)$$

we have

$$\mathcal{D}(\Phi(\omega,v) \times \Phi(\omega,v)) = \Phi(\omega,v),$$

$$N(\Phi(\omega,v) \times \Phi(\omega,v)) = \Phi(\omega,v),$$

and

$$\{y \in \mathcal{D}(\Phi(\omega,v) \times \Phi(\omega,v)) : (y,x) \notin \Phi(\omega,v) \times \Phi(\omega,v)\} = \emptyset.$$

the $h_{X\times X}$-h_X-continuity of $\mathcal{D}(\cdot)$, we have for any $y^0 \in \mathcal{D}(E^0)$ a sequence $\{y^n\}_n$ with $y^n \in \mathcal{D}(E^n)$ for all n and $y^n \underset{\rho_X}{\longrightarrow} y^0$. This, together with $E^n \underset{h_{X\times X}}{\longrightarrow} E^0$ and $x^n \underset{\rho_X}{\longrightarrow} x^0$, imply that $(y^0, x^0) \in E^0$ for any $y^0 \in \mathcal{D}(E^0)$. Thus, $(E^0, x^0) \in GrN(\cdot)$. By compactness, the fact that $GrN(\cdot)$ is closed implies that $N(\cdot)$ is upper semicontinuous - with nonempty, compact values. **Q.E.D.**

4.6 Essential Sets and the 3M Property in the Hyperspace of Ky Fan Sets

We begin by revisiting the notions of essential and minimally essential sets but for KFCs.

Definition 7 *(Again, Essential Sets and Minimal Essential Sets)*
Let $N(\cdot)$ be a KFC and let $E^0 \in \mathbb{S}$ be a given Ky Fan set.
(1) A nonempty, closed subset $e(E^0)$ of $N(E^0)$ is said to be essential for $N(\cdot)$ at $E^0 \in \mathbb{S}$ if for any $\varepsilon > 0$ there exists $\delta^\varepsilon > 0$ such that for all $E \in \mathbb{S}$ with $E \in B_{h_{X\times X}}(\delta^\varepsilon, E^0)$,

$$N(E) \cap B_{\rho_X}(\varepsilon, e(E^0)) \neq \emptyset. \tag{44}$$

We will denote by $\mathcal{E}[N(E^0)]$ the collection of all nonempty, closed subsets of $N(E^0)$ essential for $N(\cdot)$ at $E^0 \in \mathbb{S}$.
(2) A nonempty closed subset $m(E^0)$ of $N(E^0)$ is said to be minimally essential for $N(\cdot)$ at $E^0 \in \mathbb{S}$ if (i) $m(E^0) \in \mathcal{E}[N(E^0)]$ and if (ii) $m(E^0)$ is a minimal element of $\mathcal{E}[N(E^0)]$ ordered by set inclusion (i.e., if $e(E^0) \in \mathcal{E}[N(E^0)]$ and $e(E^0) \subseteq m(E^0)$ then $e(E^0) = m(E^0)$). We will denote by $\mathcal{E}^*[N(E^0)]$ the collection of all nonempty, closed subsets of $N(E^0)$ minimally essential for $N(\cdot)$ at $E^0 \in \mathbb{S}$.

Note that for any $E \in \mathbb{S}$, if B is a proper subset of $m(E)$, then $B \notin \mathcal{E}[N(E)]$. The $3M$ property for KFCs is defined as follows:

Definition 8 *(3M KFCs)*
Let $\mathcal{G}(\Omega \times V)$ be a parameterized, state contingent game satisfying assumptions [$\mathcal{PSG}$], and let, $N(\cdot) \in \mathcal{U}_{\mathbb{S}\text{-}X}$, be a KFC.
We say that $N(\cdot)$ is $3M$ at $E^0 \in \mathbb{S}$ if, given any $\delta > 0$ and given any pair of nonempty, disjoint, closed sets, F^1 and F^2 in X, there exists Ky Fan sets E^1 and E^2 in $B_{h_{X\times X}}(\delta, E^0)$ such that

$$N(E^1) \cap F^1 = \emptyset \text{ and } N(E^2) \cap F^2 = \emptyset,$$

then there exists a third Ky Fan set, E^3, in the larger open ball, $B_{h_{X\times X}}(3\delta, E^0)$, such that

$$N(E^3) \cap [F^1 \cup F^2] = \emptyset.$$

We say that the KFC, $N(\cdot)$, is $3M$ if $N(\cdot)$ is $3M$ at E for all $E \in \mathbb{S}$. We will denote by $\mathcal{U}^{3M}_{\mathbb{S}\text{-}X}$ the collection of all $3M$ KFCs.

Our next Theorem, the $3M$ Theorem, establishes that under assumptions [$\mathcal{PSG}$], $\mathcal{U}_{\mathbb{S}\text{-}X} = \mathcal{U}^{3M}_{\mathbb{S}\text{-}X}$ - all USCOs defined on the hyperspace of Ky Fan sets (i.e., all KFCs) are $3M$ and therefore all minimal USCOs defined on the hyperspace of Ky Fan sets (i.e., all minimal KFCs) take closed, connected, minimally essential Nash values.

Theorem 7 *(The $3M$ Theorem)*
Let $\mathcal{G}(\Omega \times V)$ be a parameterized, state contingent game satisfying assumptions [$\mathcal{PSG}$]. Then $\mathcal{U}_{\mathbb{S}\text{-}X} = \mathcal{U}^{3M}_{\mathbb{S}\text{-}X}$.

Proof: We have $\mathcal{U}^{3M}_{\mathbb{S}\text{-}X} \subset \mathcal{U}_{\mathbb{S}\text{-}X}$. Suppose $N(\cdot) \in \mathcal{U}_{\mathbb{S}\text{-}X}$ does not have the $3M$ property at $E^0 \in \mathbb{S}$. Then for some $\delta^0 > 0$ and some pair of closed disjoint sets F^1 and F^2 in X, the open ball, $B_{h_{X\times X}}(\delta^0, E^0) \subset \mathbb{S}$ contains two Ky Fan sets, E^1 and E^2, such that

$$N(E^1) \cap F^1 = \emptyset \text{ and } N(E^2) \cap F^2 = \emptyset, \tag{45}$$

but such that for all $E \in B_{h_{X\times X}}(3\delta^0, E^0)$, $N(E) \cap [F^1 \cup F^2] \neq \emptyset$. We will show that this leads to a contradiction by exhibiting a Ky Fan set, $E^* \in B_{h_{X\times X}}(3\delta^0, E^0)$ such that if $N(E^*) \cap [F^1 \cup F^2] \neq \emptyset$, then $N(E^i) \cap F^i \neq \emptyset$ for $i = 1$ or 2 violating (45).

First, given that the KFC $N(\cdot)$ is an USCO, under $[\mathcal{PSG}]$ there are disjoint open sets U^i such that $F^i \subset U^i$ and $N(E^i) \cap U^i = \emptyset$, $i = 1, 2$, and moreover, such that

$$\left.\begin{array}{c} N(E^*) \cap [F^1 \cup F^2] \neq \emptyset \text{ for all } E^* \in B_{h_{X\times X}}(3\delta^0, E^0) \cap \mathbb{S}, \\ \text{implies that} \\ N(E^*) \cap [U^1 \cup U^2] \neq \emptyset \text{ for all } E^* \in B_{h_{X\times X}}(3\delta^0, E^0) \cap \mathbb{S}. \end{array}\right\} \tag{46}$$

We will show that (46) leads to a contradiction by constructing a Ky Fan set, $E^* \in \mathbb{S}$ with $E^* \in B_{h_{X\times X}}(3\delta^0, E^0)$ such that

$$N(E^*) \cap [U^1 \cup U^2] \neq \emptyset \ (*),$$

implying that $N(E^i) \cap U^i \neq \emptyset$ for some $i = 1$ and/or 2. Our candidate for such a set is given by

$$E^* := [E^1 \cap (X \times U^2)^c] \cup [E^2 \cap (X \times U^1)^c] \tag{47}$$

where

$$(X \times U^i)^c := \left\{(y, x) \in X \times X : x \notin U^i\right\}.$$

To complete the proof we must show that,

$$\begin{array}{c} \text{(a) } E^* \in \mathbb{S}, \\ \text{(b) } E^* \in B_{h_{X\times X}}(3\delta^0, E^0), \\ \text{and} \\ \text{(c) } N(E^*) \cap \left[U^1 \cup U^2\right] \neq \emptyset \Rightarrow N(E^i) \cap U^i \neq \emptyset \text{ for some } i = 1, 2. \end{array}$$

(a) $E^* \in \mathbb{S}$: It is easy to see that $E^* \in P_f(X \times X)$. Moreover, because $E^i \in \mathbb{S}$ $i = 1, 2$, it is easy to see that (Z1) holds for E^*.[13] Thus, $\mathcal{D}(E^*)$ is nonempty, closed, and convex and

$$\mathcal{D}(E^*) = \mathcal{R}(E^*) = \mathcal{D}(E^0).$$

Also, it is easy to see that (Z2) holds for E^*.[14] Thus, $(y, y) \in E^*$ for all $y \in \mathcal{D}(E^*)$.

It remains to show that for all $x \in \mathcal{R}(E)$,

$$\{y \in \mathcal{D}(E^*) : (y, x) \notin E^*\}$$

is convex or empty.

Let $x \in U^1$, then because U^1 and U^2 are disjoint,

$$\{y \in \mathcal{D}(E^*) : (y, x) \notin E^*\} = \left\{y \in \mathcal{D}(E^*) : (y, x) \notin E^1\right\},$$

a convex or empty set because $E^1 \in \mathbb{S}$.

Let $x \in U^2$, then because U^1 and U^2 are disjoint,

$$\{y \in \mathcal{D}(E^*) : (y, x) \notin E^*\} = \left\{y \in \mathcal{D}(E^*) : (y, x) \notin E^2\right\},$$

[13] (Z1) $\mathcal{D}(E)$ is nonempty, closed, and convex and $\mathcal{D}(E) = \mathcal{R}(E)$.

[14] (Z2) for all $y \in \mathcal{D}(E)$, $(y, y) \in E$.

a convex or empty set because $E^2 \in \mathbb{S}$.

Let $x \in \mathcal{D}(E^*)\backslash U^1 \cup U^2$. Then

$$\{y \in \mathcal{D}(E^*) : (y,x) \notin E^*\}$$

$$= \left\{y \in \mathcal{D}(E^*) : (y,x) \notin E^1\right\} \cap \left\{y \in \mathcal{D}(E^*) : (y,x) \notin E^2\right\},$$

the later being the intersection of convex or empty sets. Therefore,

$$\{y \in \mathcal{D}(E^*) : (y,x) \notin E^*\}$$

is convex or empty.

(b) $E^* \in B_{h_{X\times X}}(3\delta^0, E^0)$: We have

$$E^* = [E^1 \cap (X \times U^2)^c] \cup [E^2 \cap (X \times U^1)^c] \tag{48}$$

and by the triangle inequality,

$$\begin{gathered} h_{X\times X}(E^1,E^2) \le h_{X\times X}(E^1,E^0) + h_{X\times X}(E^0,E^2) < 2\delta^0, \\ \text{and} \\ h_{X\times X}(E^*,E^0) \le h_{X\times X}(E^*,E^1) + h_{X\times X}(E^1,E^0). \end{gathered} \tag{49}$$

We know already that $h_{X\times X}(E^1,E^0) < \delta^0$. Consider $h_{X\times X}(E^*,E^1)$. We have

$$h_{X\times X}(E^*,E^1) := \max\left\{e_{X\times X}(E^*,E^1), e_{X\times X}(E^1,E^*)\right\}.$$

It is easy to check that,

$$e_{X\times X}(E^*,E^1) = \sup_{(y,x)\in E^*} dist_{X\times X}((y,x),E^1)$$

$$= \sup_{(y,x)\in[E^2\cap(X\times U^1)^c]} dist_{X\times X}((y,x),E^1)$$

$$\le \sup_{(y,x)\in E^2} dist_{X\times X}((y,x),E^1) = e_{X\times X}(E^2,E^1).$$

To show that $e_{X\times X}(E^1,E^*) \le e_{X\times X}(E^1,E^2)$ observe that

$$e_{X\times X}(E^1,E^*) = \sup_{(y,x)\in E^1} dist_{X\times X}((y,x),E^*)$$

$$= \sup_{(y,x)\in E^1} dist_{X\times X}((y,x),[E^1\backslash(X\times U^2)] \cup [E^2\backslash(X\times U^1)]).$$

Letting $E^1 = [E^1\backslash(X\times U^2)] \cup \left[E^1 \cap (X\times U^2)\right]$, we have for all

$$(y,x) \in E^1\backslash(X\times U^2),$$

$$dist_{X\times X}((y,x),E^*)$$

$$= dist_{X\times X}((y,x),[E^1\backslash(X\times U^2)] \cup [E^2\backslash X\times U^1)])$$

$$\le dist_{X\times X}((y,x),[E^2\backslash(X\times U^1)] \cup [E^2 \cap (X\times U^1)])$$

$$= dist_{X\times X}((y,x),E^2).$$

Moreover, we have for all

$$(y,x) \in E^1 \cap (X\times U^2),$$

$$dist_{X\times X}((y,x),E^*)$$

$$= dist_{X\times X}((y,x),[E^1\backslash(X\times U^2)]\cup[E^2\backslash(X\times U^1)])$$

$$= dist_{X\times X}((y,x),[E^2\backslash(X\times U^1)]),$$

and

$$dist_{X\times X}((y,x),E^2)$$

$$= dist_{X\times X}((y,x),[E^2\backslash(X\times U^1)]\cup[E^2\cap(X\times U^1)])$$

$$= dist_{X\times X}((y,x),[E^2\backslash(X\times U^1)]).$$

Thus, for all $(y,x)\in E^1$,

$$\rho_{X\times X}((y,x),E^*)\leq \rho_{X\times X}((y,x),E^2),$$

implying that $e_{X\times X}(E^1,E^*)\leq e_{X\times X}(E^1,E^2)$. Together,

$$e_{X\times X}(E^1,E^*)\leq e_{X\times X}(E^1,E^2)$$
and
$$e_{X\times X}(E^*,E^1)\leq e_{X\times X}(E^2,E^1)$$

imply that

$$h_{X\times X}(E^*,E^1)\leq h_{X\times X}(E^2,E^1)<2\delta^0.$$

Thus, we have

$$h_{X\times X}(E^*,E^0)\leq h_{X\times X}(E^*,E^1)+h_{X\times X}(E^1,E^0)$$

$$\leq h_{X\times X}(E^2,E^1)+h_{X\times X}(E^1,E^0)$$

$$< 2\delta^0+\delta^0=3\delta^0.$$

(c) $N(E^*)\cap\left[U^1\cup U^2\right]\neq\emptyset\Rightarrow N(E^i)\cap U^i\neq\emptyset$ for some $i=1$ and/or 2:

WLOG suppose that $x\in N(E^*)\cap U^1$. Given the definition of the KFC, $N(\cdot)$, We have for each $x\in N(E^*)$ and $y\in\mathcal{D}(E^*)$,

$$(y,x)\in\left(E^1\cap\left(X\times U^2\right)^c\right)\cup\left(E^2\cap\left(X\times U^1\right)^c\right),$$

and because $x\in U^1$, this implies that for each $y\in\mathcal{D}(E^*)$,

$$(y,x)\in E^1\cap\left(X\times U^2\right)^c,$$

and specifically, that for each $y\in\mathcal{D}(E^*)$,

$$(y,x)\in E^1\cap\left(X\times U^1\right).\ (*)$$

Thus, given that $x\in N(E^*)$ and $y\in\mathcal{D}(E^*)$, (*) implies that

$$x\in N(E^1)\cap U^1,$$

contradicting the fact that $N(E^1)\cap U^1=\emptyset$. Thus we must conclude that $N(\cdot)$ has the 3M property. **Q.E.D.**

The following result concerning the equivalence of $3M$ and connectedness for minimal KFCs is an immediate consequence of Theorem 4 above.

Theorem 8 *(For Minimal KFCs being Minimally Essential and Continuum Valued is Equivalent to Being $3M$)*
Let $\mathcal{G}(\Omega \times V)$ be a parameterized, state contingent game satisfying assumptions [$\mathcal{PSG}$] and let $N(\cdot) \in \mathcal{U}_{\mathbb{S}\text{-}X}$ be a KFC. The following statements about $n(\cdot) \in [N(\cdot)]$ are equivalent:
(1) $n(\cdot)$ is $3M$ at $E^0 \in \mathbb{S}$.
(2) $n(E^0)$ is connected.

5 The Collective Security Mapping - the CSM

With each (ω, v)-game,

$$\mathcal{G}_{(\omega,v)} := \{\Phi_d(\omega, v_d), u_d(\omega, v_d, (\cdot, \cdot))\}_{d \in D}\,, \tag{50}$$

we can associate a *Nikaido-Isoda function* (Nikaido and Isoda, 1955) given by

$$\left.\begin{aligned} &\varphi(\omega, v, (y, x)) := u(\omega, v, (y, x)) - u(\omega, v, (x, x)) \\ &:= \textstyle\sum_{d \in D} u_d(\omega, v_d, (y_d, x_{-d})) - \sum_{d \in D} u_d(\omega, v_d, (x_d, x_{-d})), \end{aligned}\right\} \tag{51}$$

for each $(y, x) \in \Phi(\omega, v) \times \Phi(\omega, v)$. We say that $x' \in \Phi(\omega, v)$ is *collectively secure against a feasible defection profile, $y' \in \Phi(\omega, v)$,* with player specific noncooperative player defections given by, (y'_d, x'_{-d}), for players $d = 1, 2, \ldots, m$, if and only if

$$(y', x') \in K(\omega, v) := \{(y, x) \in \Phi(\omega, v) \times \Phi(\omega, v) : \varphi(\omega, v, (y, x)) \leq 0\}.$$

Thus, for each one-shot game, $\mathcal{G}(\Omega \times V)$, satisfying assumptions [$\mathcal{PSG}$], there is collective security mapping - a CSM - given by

$$(\omega, v) \longrightarrow K(\omega, v) := \{(y, x) \in \Phi(\omega, v) \times \Phi(\omega, v) : \varphi(\omega, v, (y, x)) \leq 0\}. \tag{52}$$

The *collectively secure action mapping* (i.e., the CS action mapping) is given by,

$$y \longrightarrow K(\omega, v)(y) := \{x \in \Phi(\omega, v) : \varphi(\omega, v, (y, x)) \leq 0\}\,. \tag{53}$$

For each defection profile $y \in \Phi(\omega, v)$ with player specific defections of the form $y = (y_d, x_{-d})$, $K(\omega, v)(y)$ is the (closed) set of action profiles, $x = (x_d, x_{-d})$, in $\Phi(\omega, v)$ that are *collectively secure* against potential *noncooperative* defections represented by profile, y. Note that if x is contained in $K(\omega, v)(y)$ for *all* possible defection profiles $y \in \Phi(\omega, v)$, that is, if

$$x \in \cap_{y \in \Phi(\omega,v)} K(\omega, v)(y) \tag{54}$$

then for each player d, $x = (x_d, x_{-d})$ is secure against any defection of the form $y = (y_d, x_{-d})$. Thus, $x \in \cap_{y \in \Phi(\omega,v)} K(\omega, v)(y)$ implies that

$$u_d(\omega, v_d, (y_d, x_{-d})) \leq u_d(\omega, v_d, (x_d, x_{-d})),$$

for all players, d, and all pairs $y = (y_d, x_{-d})$ and $x = (x_d, x_{-d})$ - and conversely. Thus, the set of Nash equilibria given state-value function profile pair (ω, v) can be fully characterized as follows:

$$x \in \mathcal{N}(\omega, v) \text{ if and only if } x \in \cap_{y \in \Phi(\omega,v)} K(\omega, v)(y), \tag{55}$$

and therefore, the Nash correspondence is given by,

$$(\omega, v) \longrightarrow \mathcal{N}(\omega, v) = \cap_{y \in \Phi(\omega,v)} K(\omega, v)(y). \tag{56}$$

Under assumptions [$\mathcal{PSG}$], the function, $\varphi(\cdot, \cdot, (\cdot, \cdot))$ which specifies for each $(\omega, v) \in \Omega \times V$ a particular Nikaido-Isoda function has the following properties:

(F1) for each ω, $\varphi(\omega, \cdot, (\cdot, \cdot))$ is continuous on the compact metric space, $V \times (X \times X)$;

(F2) for each $(v, (y, x))$, $\varphi(\cdot, v, (y, x))$ is (B_Ω, B_Y)-measurable; and

(F3) $y \longrightarrow \varphi(\omega, v, (\cdot, x))$ is concave in y on X.

For each state-value function profile pair (ω, v), the graph of the CS action mapping, $K(\omega, v)(\cdot)$, is given by

$$GrK(\omega, v)(\cdot) := \{(y, x) \in \Phi(\omega, v) \times \Phi(\omega, v) : \varphi(\omega, v, (y, x)) \leq 0\}. \tag{57}$$

Thus, for any $(y', x') \in GrK(\omega, v)(\cdot)$, strategy profile $x' \in \Phi(\omega, v)$ is secure against defection profile $y' \in \Phi(\omega, v)$ and we have $\varphi(\omega, v, (y', x')) \leq 0$. *Thus, for each $(\omega, v) \in \Omega \times V$, the value of the CSM, $K(\omega, v)$, is given by the graph of the CS action mapping $K(\omega, v)(\cdot)$ - and we will show that $K(\omega, v) = GrK(\omega, v)(\cdot)$ is a Ky Fan set. Thus, the CSM is given by the Ky Fan set-valued mapping,*

$$(\omega, v) \longrightarrow K(\omega, v) := GrK(\omega, v)(\cdot) \in \mathbb{S} \text{ for all } (\omega, v) \in \Omega \times \mathcal{L}_Y^\infty. \tag{58}$$

Moreover, we will show that for each minimal KFC, $n(\cdot)$, the composition correspondence, $(\omega, v) \longrightarrow n(K(\omega, v))$, is upper Caratheodory and takes connected values.

Our main results regarding the collective security mapping are the following:

Theorem 9 *(The collective security function, $K(\cdot, \cdot)$, is Ky Fan valued and Caratheodory)*
Let $\mathcal{G}(\Omega \times V)$ be a parameterized, state contingent game satisfying assumptions [$\mathcal{PSG}$] with Nash correspondence, $\mathcal{N}(\cdot, \cdot) \in \mathcal{UC}_{\Omega \times V\text{-}P_f(X)}$, KFC, $N(\cdot) \in \mathcal{U}_{\mathbb{S}\text{-}X}$. and CSM, $K(\cdot, \cdot)$. Then the following statements about the CSM, $K(\cdot, \cdot)$, are true:
(1) For each state-parameter pair, $(\omega, v) \in \Omega \times V$, $K(\omega, v)$ is a Ky Fan set with

$$\mathcal{D}(K(\omega, v)) = \Phi(\omega, v) \text{ and } \mathcal{R}(K(\omega, v)) = \Phi(\omega, v).$$

(2) For each minimal KFC, $n \in [N]$, $n(K(\cdot, \cdot))$, is upper Caratheodory (i.e., $n(K(\cdot, \cdot))$ is $(B_\Omega \times B_V\text{-}B_X)$-measurable and for each ω, $n(K(\omega, \cdot))$ is ρ_V-ρ_X-upper semicontinuous (i.e., for each ω, $v^n \underset{\rho_V}{\longrightarrow} v$ and $x^n \underset{\rho_X}{\longrightarrow} x$ with $x^n \in n(K(\omega, v^n))$ for all n implies that $x \in n(K(\omega, v)))$.

Proof of (1): It is easy to see that $\mathcal{D}(K(\omega, v)) = \mathcal{R}(K(\omega, v)) = \Phi(\omega, v)$. Thus (Z1) holds. Let $(\omega, v) \in \Omega \times V$. We must show that $K(\omega, v) \in \mathbb{S}$. Recall that

$$K(\omega, v) := \{(y, x) \in \Phi(\omega, v) \times \Phi(\omega, v) : \varphi(\omega, v, (y, x)) \leq 0\},$$

and note that for all $y \in \Phi(\omega, v)$, $\varphi(\omega, v, (y, y)) = 0$. Thus, (Z2) holds.

To see that (Z3) holds observe that because $\varphi(\omega, v, (\cdot, x))$ is affine in y, $y \in \Phi(\omega, v)$ such that $(y, x) \notin K(\omega, v)$ is given by the set, $\{y \in \Phi(\omega, v) : \varphi(\omega, v, (y, x)) > 0\}$, and this set is convex (or empty).

Proof of (2): Let ω be fixed and suppose that the sequence, $\{(v^n, x^n)\}_n$, is such that, $v^n \underset{\rho_V}{\longrightarrow} v^0$ and $x^n \underset{\rho_X}{\longrightarrow} x^0$ with $x^n \in n(K(\omega, v^n))$ for all n. By the continuity of $\Phi(\omega, \cdot)$, we have for any $y^0 \in \Phi(\omega, v^0)$ a sequence $\{y^n\}_n$ with $y^n \in \Phi(\omega, v^n)$ for all n such that $y^n \underset{\rho_X}{\longrightarrow} y^0$, and because $x^n \in n(K(\omega, v^n))$ for all n, we have that $\varphi(\omega, v^n, (y^n, x^n)) \leq 0$ for all n. Thus by the continuity of $\varphi(\omega, \cdot, (\cdot, \cdot))$ for each ω, we have in the limit that $\varphi(\omega, v^0, (y^0, x^0)) \leq 0$ for any point $y^0 \in \Phi(\omega, v^0)$ implying that $x^0 \in n(K(\omega, v^0))$.

That $n(K(\cdot, \cdot))$ is $(B_\Omega \times B_V\text{-}B_X)$-measurable follows from Lemma 3.1 in Kucia and Nowak (2000).**Q.E.D.**

5.1 Parameter Values, Ky Fan Sets, and Nash Equilibria

For the collection of (ω, v)-games (i.e., the parameterized state-contingent game), $\mathcal{G}(\Omega \times V)$, satisfying $[\mathcal{PSG}]$, the Nash correspondence is given by

$$\mathcal{N}(\omega, v) = N(K(\omega, v)) := N \circ K(\omega, v) \text{ for all } (\omega, v) \in \Omega \times V, \tag{59}$$

where the Ky Fan valued CSM, $K(\cdot,\cdot)$, is such that for each minimal KFC, $n(\cdot)$, belonging to $N(\cdot)$, the composition correspondence, $(\omega, v) \longrightarrow n(K(\omega, v))$, is upper Caratheodory and takes connected values with

$$n(K(\omega, v)) \subset N(K(\omega, v)) = \mathcal{N}(\omega, v) \text{ for all } (\omega, v) \in \Omega \times V.$$

Thus, for the KFC, $N(\cdot)$, we have for each Ky Fan set $E \in \mathbb{S}$ that

$$N(E) = \cap_{y \in \mathcal{D}(E)} \left\{ x \in \mathcal{R}(E) : (y, x) \in E \right\}. \tag{60}$$

Thus, the Nash correspondence given as the composition of the CSM, $K(\cdot,\cdot)$, and the KFC, $N(\cdot)$, is given by

$$\left.\begin{aligned} \mathcal{N}(\omega, v) = N(K(\omega, v)) &= \cap_{y \in \mathcal{D}(K(\omega, v))} \left\{ x \in \mathcal{R}(K(\omega, v)) : (y, x) \in K(\omega, v) \right\}, \\ &= \cap_{y \in \Phi(\omega, v)} \underbrace{\{ x \in \Phi(\omega, v) : \varphi(\omega, v, (y, x)) \le 0 \}}_{K(\omega, v)(y)}, \end{aligned}\right\} \tag{61}$$

where $K(\omega, v) \in \mathbb{S}$.
Because all minimal USCOs, $n(\cdot)$, belonging to a KFC, $N(\cdot)$, (i.e., all minimal KFCs) *are continuum-valued* and because the CSM, $K(\cdot,\cdot)$, belonging to the parameterized game, $\mathcal{G}(\Omega \times V)$, is a Ky Fan valued, the composition mapping

$$v \longrightarrow n(K(\omega, v))$$

is a continuum-valued minimal ω-USCO for the Nash ω-USCO, $\mathcal{N}(\omega, \cdot)$ (i.e., $n(K(\omega, \cdot)) \in [\mathcal{N}(\omega, \cdot)]$ with continuum values). Thus, we have for any parameterized game, $\mathcal{G}(\Omega \times V)$, satisfying $[\mathcal{PSG}]$ with Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, CSM, $K(\cdot,\cdot)$, and KFC, $N(\cdot)$, that

$$n(K(\cdot,\cdot)) \in \mathcal{MUC}^{\mathcal{N}}_{C_f(X)}.$$

where $C_f(X)$ is h_X-compact hyperspace of continua - an h_X-closed sub-hyperspace of $P_f(X)$. Because $n(\cdot)$ is a *minimal KFC and therefore* $3M$, $n(\cdot)$ is not only minimally essential valued but also *connected-valued* (see Theorem 8 above). What we conclude from all of this is that if $n(\cdot)$ is a minimal USCO for the KFC $N(\cdot)$, then $n(K(\cdot,\cdot))$ is a continuum valued, minimal upper Caratheodory mapping for the Nash correspondence, $\mathcal{N}(\cdot,\cdot)$. Formally, we have the following result.

Theorem 10 *(The Composition Theorem: A minimal KFC composed with the CSM is a continuum-valued minimal Nash correspondence)*
Suppose the parameterized state-contingent game, $\mathcal{G}(\Omega \times V)$, satisfies assumption $[\mathcal{PSG}]$ with upper Caratheodory Nash correspondence, $\mathcal{N}(\cdot,\cdot) = N(K(\cdot,\cdot))$ where $N(\cdot) \in \mathcal{U}(\mathbb{S}, P_f(X))$ is the KFC and $K(\cdot,\cdot)$, a set-valued correspondence taking Ky Fan values defined on $\Omega \times V$ is the CSM. Then for each $n(\cdot) \in [N(\cdot)]$

$$n(K(\omega, \cdot)) \in [\mathcal{N}(\omega, \cdot)] \text{ for each } \omega.$$

Proof: Suppose that for some ω^0, $n(K(\omega^0,\cdot))$ is not a minimal USCO belonging to $\mathcal{N}(\omega^0,\cdot)$. Let $m_{\omega^0}(\cdot)$ be a minimal USCO of $n(K(\omega^0,\cdot)) \in \mathcal{U}(V, P_f(X))$ such that for some $v^0 \in V$, $m_{\omega^0}(v^0)$ is a proper subset of $n(K(\omega^0, v^0))$ (i.e., an USCO defined on V with values in $P_f(X)$ such that $m_{\omega^0}(\cdot) \in [n(K(\omega^0,\cdot))]$). Because $m_{\omega^0}(\cdot) \in [n(K(\omega^0,\cdot))]$, we have for each $\varepsilon^0 > 0$ a $\delta^0 > 0$ such that for all

$$\left.\begin{aligned} v^{\delta^0} \in B_{\rho_V}(\delta^0, v^0) \subset V, \\ n(K(\omega^0, v^{\delta^0})) \cap B_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0)) \neq \emptyset. \end{aligned}\right\} \tag{62}$$

Because

$$n(K(\omega^0, v^{\delta^0})) \cap B_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0))$$

is a closed subset of both $n(K(\omega^0, v^{\delta^0}))$ and $\overline{B}_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0))$,

$$n(K(\omega^0, v^{\delta^0})) \cap \overline{B}_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0))$$

is not an essential set of $n(K(\omega^0, v^{\delta^0}))$ in V. Therefore, there is some $\varepsilon^1 > 0$ such that for each n there exists $v^{\delta^n} \in B_{\rho_V}(\frac{1}{n}, v^{\delta^0}) \subset V$ such that

$$n(K(\omega^0, v^{\delta^n})) \cap B_{\rho_X}\left[\varepsilon^1, n(K(\omega^0, v^{\delta^0})) \cap \overline{B}_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0))\right] = \emptyset. \tag{63}$$

Given (62) and the fact that for n sufficiently large

$$v^{\delta^n} \in B_{\rho_V}(\tfrac{1}{n}, v^{\delta^0}) \subset B_{\rho_V}(\delta^0, v^0) \subset V,$$

we have

$$n(K(\omega^0, v^{\delta^n})) \cap \overline{B}_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0)) \neq \emptyset.$$

Therefore, for all n sufficiently large, we have some

$$x^n \in n(K(\omega^0, v^{\delta^n})) \cap \overline{B}_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0)).$$

Because $x^n \in n(K(\omega^0, v^{\delta^n}))$, for all n sufficiently large we have by (63),

$$x^n \notin B_{\rho_X}\left[\varepsilon^1, n(K(\omega^0, v^{\delta^0})) \cap \overline{B}_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0))\right]. \tag{64}$$

WLOG, suppose that $x^n \underset{\rho_X}{\longrightarrow} x^0$ and note that $v^{\delta^n} \underset{\rho_V}{\longrightarrow} v^{\delta^0}$. Thus, because $n(K(\omega^0,\cdot))$ is an USCO and $\overline{B}_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0))$ is closed, we have

$$x^0 \in n(K(\omega^0, v^{\delta^0})) \cap \overline{B}_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0)).$$

But now we have a contradiction, because by (64) it must be the case that for some $\varepsilon^2 \in (0, \varepsilon^1)$,

$$x^0 \notin B_{\rho_X}\left[\varepsilon^2, n(K(\omega^0, v^{\delta^0})) \cap \overline{B}_{\rho_X}(\varepsilon^0, m_{\omega^0}(v^0))\right].$$

Therefore, we must conclude that for no $v \in V$ can it be true that $m_{\omega^0}(v)$ is a proper subset of $n(K(\omega^0, v))$ - and therefore we can conclude that $n(K(\omega^0,\cdot)) \in [\mathcal{N}(\omega^0,\cdot)]$. **Q.E.D.**

5.2 Comments

(1) For the Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, belonging to any parameterized state contingent game, $\mathcal{G}(\Omega\times V)$, satisfying assumptions $[\mathcal{PSG}]$, we know that $\mathcal{MUC}^{\mathcal{N}}\neq\varnothing$. In particular, we have $n(K(\cdot,\cdot))\in\mathcal{MUC}^{\mathcal{N}}$. But more importantly, we now know that for each ω, $n(K(\omega,\cdot))$ is minimally essential and continuum-valued. Therefore, by Theorem 8, we know that for each ω, $n(K(\omega,\cdot))$ is $3M$. Thus, we have that

$$n(K(\cdot,\cdot))\in\mathcal{MUC}^{\mathcal{N}}_{C_f(X)}\subset\mathcal{UC}_{\Omega\times V\text{-}P_f(X)}.$$

Note that if the KFC, $N(\cdot)$ is quasi-minimal so that $[N(\cdot)]=\{n(\cdot)\}$ for some minimal KFC, then

$$\{n(K(\cdot,\cdot))\}=\mathcal{MUC}^{\mathcal{N}}.$$

(2) Let

$$K_\omega(V):=\cup_{v\in V}K_\omega(v)$$

denote the range of the function, $v\longrightarrow K(\omega,v)$, from the set of parameter profiles, V, into Ky Fan sets, $\mathbb{S}$. Because V is ρ_V-compact and $K_\omega(\cdot)$ is ρ_V-ρ_X-upper semicontinuous for each ω, $K_\omega(V)$ is ρ_X-compact. Moreover, because V is locally connected, if for each ω, $v\longrightarrow K(\omega,v)$ is ρ_V-ρ_X-lower semicontinuous, then $v\longrightarrow K(\omega,v)$ is ρ_V-$h_{\mathbb{S}}$-continuous and $K_\omega(V)$ is locally connected - implying that because $K_\omega(V)$ is also connected, for each ω, $K_\omega(V)$ is a Peano continuum. Therefore, we can assume without loss of generality that the sub-hyperspace of Ky Fan sets, $K_\omega(V)$, specific to a particular parameterized game, $\mathcal{G}(\Omega\times V)$, can be equipped with an M-convex metric, $h_{K_\omega(V)}$ equivalent to the metric $h_{\mathbb{S}}$ restricted to $K_\omega(V)$. Thus, for any two distinct Ky Fan sets E^1 and E^2 in $K_\omega(V)$ there is a third Ky Fan set $\overline{E}\in K_\omega(V)$ such that

$$h_{K_\omega(V)}(E^1,E^2)=h_{K_\omega(V)}(E^1,\overline{E})+h_{K_\omega(V)}(\overline{E},E^2).$$

Moreover, by Theorem 2.7 in Nadler (1977) for any two distinct Ky Fan sets E^1 and E^2 in $K_\omega(V)$ there is a subset $\gamma\subset K_\omega(V)\subset\mathbb{S}$ such that $E^1\in\gamma$ and $E^2\in\gamma$ where γ is isometric to the interval $[0,h_{K_\omega(V)}(E^1,E^2)]$ and such that if $E^1=E^2$ then $\gamma=\{E^1\}=\{E^2\}$ and if $E^1\neq E^2$, then γ is an arc with end points E^1 and E^2.

(3) Page (2015a) shows very directly that if there is a Nash sub-correspondence $\eta(\cdot,\cdot)\in\mathcal{UC}^{\mathcal{N}}$ taking contractibly-values or R_δ-values in X, then the Nash payoff selection correspondence will have fixed points. But it is difficult to identify conditions on primitives guaranteeing that Nash sub-correspondences, $\eta(\cdot,\cdot)\in\mathcal{UC}^{\mathcal{N}}$, take contractibly values or R_δ-values in X. Here we have shown that for parameterized, state-contingent games satisfying assumptions $[\mathcal{PSG}]$, it is automatic that all such $\mathcal{PSG}s$ possess uC Nash correspondence having continuum valued minimal uC Nash correspondences. This in turn implies that in all such games players have induced uC Nash payoff sub-correspondences taking interval values and hence contractible values - further implying that all such games have Nash payoff selection correspondences having fixed points.

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