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Keywords: stationary Markov equilibria, discounted stochastic games, fixed point theorem for nonconvex, measurable-selection-valued correspondences, Komlos convergence, weak star convergence, sub-USCOs, contractible values, R_δ-values, continuous approximation

JEL Classification: C7

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Stationary Markov Equilibria for Approximable Discounted Stochastic Games

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Abstract

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1 Introduction

The existence or nonexistence of stationary Markov equilibria for *uncountable-compact* discounted stochastic games was an open question from the time of the Himmelberg, Parthasarathy, Raghavan, and Van Vleck paper on p-equilibria in stationary strategies in 1976 until 2013.[1] Then came the papers of Levy (2013) and Levy and McLennan (2014), which essentially settled the matter in the negative: uncountable-compact discounted stochastic games do not always have stationary Markov equilibria. The cause of the nonexistence problem: Nash equilibria homeomorphic to the unit circle. The purpose of this paper is to move the boundary of the existence/nonexistence literature in the positive direction by first identifying a new large class of uncountable-compact discounted stochastic games for which existence can be established and by then establishing two new existence results for this class. We call this class of discounted stochastic games (DSGs) the *approximable class*.[2] The class of approximable DSGs includes supermodular DSGs and DSGs having weakly interacting players (Amir 1991, Curtat 1996, and Horst 2005), the new class of DSGs analyzed by Nowak (2003, 2007), the class of risky (or noisy) DSGs recently studied by Duggan (2012), and the class of all $\mathcal{G}$-nonatomic DSGs recently studied by He and Sun (2015). Because the condition which guarantees approximability - the presence of sub-correspondences taking contractible values (or more generally, R_δ-values) - is the very condition which rules out Nash equilibria homeomorphic to the unit circle, we conjecture that for uncountable-compact discounted stochastic games, the approximable class is the largest class for which existence of stationary Markov equilibria can be established.

1.1 A Brief Overview of the Literature

Stochastic games were introduced by Shapley (1953) who gave the first existence proof of stationary equilibria for two-person, discounted stochastic games with finite state spaces and finite strategy sets. Shapley's result was then extended to the n-player case by Fink (1964) - who allowed for infinite strategy spaces - and Takahashi (1964). Later other proofs appeared due to Rogers (1969) and Sobel (1971). These early existence results were then extended to discounted stochastic games with countably infinite state spaces by Parthasarathy (1973), and then by Sobel (1973), but Sobel also extended his results to games with uncountably many states. This branch of the literature seems to have culminated in the paper by Federgruen (1978) who pointed out flaws in the existence proof of Sobel (1973) for games with uncountable state spaces and who established the existence of stationary Markov equilibria for discounted stochastic games with countably infinite state spaces and compact strategy spaces.

Uncountable-compact discounted stochastic games were first analyzed in the mathematics literature for stationary equilibria by Himmelberg, Parthasarathy, Raghavan, and Van Vleck (1976) who proved, under separability conditions on state transitions and immediate payoff functions, that such games admit p-equilibria in stationary strategies.[3] Later Parthasarathy (1982) and Nowak (1987) strengthened these p-equilibria results for

[1] We will often refer to a finite-player, nonzero-sum discounted stochastic game in which players' strategy sets are compact metric spaces and the state space is uncountable as an *uncountable-compact* discounted stochastic game.

[2] We say that a discounted stochastic game is approximable if the underlying one-shot game either (i) has an upper Caratheodory Nash equilibrium correspondence whose USCO part can be graphically approximated by continuous functions (see, for example, Cellina, 1969, and De Blasi and Myjak, 1986) or (ii) has a Nash payoff selection correspondence (a measurable-selection-valued correspondence) that can be graphically approximated by continuous, measurable-selection-valued functions.

[3] A p-equilibrium in stationary strategies is a Nash equilibrium in stationary strategies for p-almost every initial state where p is probability measure p on the underlying state space.

separable models to Nash equilibria results. Weakening the separability conditions, Rieder (1979), Whitt (1980), and Nowak (1985) showed that such games admit ε-Nash equilibria in stationary Markov strategies. Then, Nowak and Raghavan (1992) and later Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) established that m-player, uncountable-compact discounted stochastic games naturally possess correlated stationary Markov equilibria, where each such equilibrium is specified by a public randomization device over a set of $m+1$ stationary Markov player strategy m-tuples. The work by Nowak and Raghavan (1992) and Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) made clear the critical importance of public randomization devices in overcoming the difficult problems of lack closedness (in the appropriate topology) and convexity of the Nash payoff selection correspondence in establishing the existence of correlated stationary Markov equilibria for games with uncountable state spaces and uncountable (but compact) strategy spaces.[4] Moreover, the analysis of Nowak and Raghavan (1992) pointed the way to various types of specializations of the discounted stochastic game model and in particular, led to the identification of special classes of discounted stochastic game models for which the existence of stationary Markov equilibria could be established (see Nowak, 2003 and 2007).

Our approach to proving existence - via continuous approximation methods - is new. In particular, for our first result we use approximation methods involving payoff-selection-valued continuous functions to establish a new fixed point result for Nash payoff selection correspondences - and more generally for measurable-selection-valued correspondences having nonconvex values. For our second result, we again use approximation methods, but this time involving player action-profile-valued continuous functions to establish a new measurable selection result (implying our second fixed point result) for upper Caratheodory Nash payoff correspondences. Because conditions which guarantee approximability - the presence of sub-correspondences taking contractible values (or more generally, R_δ-values) - are the very conditions which rule out Nash equilibria homeomorphic to the unit circle - we conjecture that for uncountable-compact discounted stochastic games, the approximable class may be the largest class for which existence of stationary Markov equilibria can be established. Moreover, we note that with regard to our first approach (approximation of the Nash payoff selection correspondence by payoff-selection-valued continuous functions) the sufficiency condition of He and Sun (2015) (i.e., the coarser transition kernel condition), as well as the sufficiency condition of Duggan (2012) (i.e., the presence of a nonatomic noisy states), imply our sufficiency condition (the K-limit property - implying approximability) for the existence of stationary Markov equilibria. Finally, we note that we carry out our analysis within the context a discounted stochastic game model similar to the models used by Mertens and Parthasarathy (1987, 1991), Nowak and Raghavan (1992), Duffie, Geanakoplos, Mas-Colell, and McLennan (1994), Salon (1998), and Maitra and Sudderth (2007) - as well as the models of He and Sun (2015) and Duggan (2012).

1.2 An Overview of the Existence Problem

An m-player discounted stochastic game can be thought of as m interdependent discounted stochastic dynamic programming problems. As a consequence, Blackwell's Theorem (1965) provides the key piece of the puzzle which leads to a solution of the existence problem in discounted stochastic games by giving necessary and sufficient conditions for existence in terms of a parameterized collection of state-dependent, one-shot games. Given state ω, this collection, parameterized by the profile of player value functions, v, is given

[4] In fact, Forges (1986) was the first to make clear the critical importance of communication and extensive form correlation devices in resolving existence issues in multistage games.

by

$$\{\mathcal{G}(\omega, v)\}_{v\in\mathcal{L}^\infty_X} := \left\{(\Phi_d(\omega), U_d(\omega, \cdot, v_d))_{d\in D}\right\}_{v\in\mathcal{L}^\infty_X}.$$

Here, D is a finite set of players who in each state ω, choose a feasible action, a_d, from a feasible set of actions, $\Phi_d(\omega)$, so as to maximize their payoff in $X_d := [-M, M]$ given by payoff function

$$a_d \longrightarrow U_d(\omega, a_d, a_{-d}, v_d) \in X_d,$$

where $\omega \in \Omega$ is the current state, $v_d \in \mathcal{L}^\infty_{X_d}$ is player $d's$ value function and a_{-d} is other players' actions.

In choosing actions through discrete time, player d uses continuation values (or state-contingent prices) or value function $v_d \in \mathcal{L}^\infty_{X_d}$ to evaluate the future consequences of current actions. Thus, the collection of one-shot, m-player games, $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^\infty_X}$, is a collection of games parameterized by the set of states and value function profiles or value function m-tuples, $v := (v_d)_{d\in D} \in \mathcal{L}^\infty_X$, where

$$X := \prod_d X_d \text{ and } \mathcal{L}^\infty_X := \prod_d \mathcal{L}^\infty_{X_d}.$$

The basic idea is the following: if in all periods players use valuations v' to price the future consequences of their immediate (current period) choices of a feasible action profile, and if in the current period state $\omega \in \Omega$ prevails, then players will arrive at a profile of immediate actions by playing the one-shot game given by

$$\mathcal{G}(\omega, v') := (\Phi_d(\omega), U_d(\omega, \cdot, v'_d))_{d\in D}.$$

Thus, in state ω, each player, d, will chose a feasible, immediate action so as to solve the problem,

$$\max_{a''_d\in\Phi_d(\omega)} U_d(\omega, a''_d, a_{-d}, v'_d).$$

An immediate action profile $a^* := (a^*_d)_{d\in D} \in \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega)$ is a Nash equilibrium for the one-shot game $\mathcal{G}(\omega, v')$ if for each player d,

$$U_d(\omega, a^*_d, a^*_{-d}, v'_d) = \max_{a_d\in\Phi_d(\omega)} U_d(\omega, a_d, a^*_{-d}, v'_d).$$

Throughout the paper we will denote by $\mathcal{N}(\omega, v)$ the set of all Nash equilibria of the one-shot game $\mathcal{G}(\omega, v)$, and by $\mathcal{P}(\omega, v)$ the corresponding set of Nash payoffs.[5] Thus, $U = (U_d)_{d\in D} \in \mathcal{P}(\omega, v)$ if and only if,

$$U = (U_1(\omega, a, v_1), \ldots, U_m(\omega, a, v_m)) \text{ for some } a \in \mathcal{N}(\omega, v).$$

The question is: which game in the collection is the "correct" game to play? Or equivalently, which value function profile should players use in pricing their action choices if each player wishes to maximize the sum of his discounted future payoffs in equilibrium?

[5] Also, we will refer to the mapping

$$(\omega, v) \to \mathcal{N}(\omega, v)$$

as the upper Caratheodory Nash mapping (measurable in ω and upper semicontinuous in v) with USCO part,

$$\{\mathcal{N}(\omega, \cdot) : \omega \in \Omega\},$$

because for each ω, $v \longrightarrow \mathcal{N}(\omega, v)$ is upper semicontinuous with nonempty compact values - i.e., $v \longrightarrow \mathcal{N}(\omega, v)$ is an USCO. Similarly, for the upper Caratheodory Nash payoff mapping,

$$(\omega, v) \to \mathcal{P}(\omega, v)$$

with USCO part,

$$\{\mathcal{P}(\omega, \cdot) : \omega \in \Omega\}.$$

If for each player d we were to (measurably) string together the single-period, immediate actions chosen by that player in each possible state ω' while playing the one shot game, $\mathcal{G}(\omega', v)$, then we would have constructed, for value function profile $v := (v_d)_{d\in D}$, a profile of (measurable) functions,

$$a(\cdot) := (a_d(\cdot))_{d\in D},$$

such that for each state ω', $a(\omega')$ is a Nash equilibrium of the one-shot game, $\mathcal{G}(\omega', v)$. And if we were to carry out this exercise using the "correct" value functions, v, then by Blackwell's Theorem for dynamic programming extended to stochastic games, we would have solved our existence problem. Thus, if the primitives of our model are such that we can always find a Nash equilibrium of our one-shot game for each value function profile, then the problem of proving the existence of stationary Markov equilibria, reduces to finding the correct value function profile - i.e., the correct prices or the correct continuation values.

Suppose that under the primitives of our model the one-shot game, $\mathcal{G}(\omega, v)$, always has a nonempty set of Nash equilibria, $\mathcal{N}(\omega, v)$, in all states ω and for all value function profiles v, and therefore, always has a nonempty set of Nash payoffs, $\mathcal{P}(\omega, v)$. Now observe that if for a given value function profile, v, the profile of real-valued (measurable) payoff functions, $(U_{(\cdot)d})_{d\in D}$, is such that in all states ω,

$$(U_{\omega d})_{d\in D} := U_\omega \in \mathcal{P}(\omega, v),$$

then $(U_{(\cdot)d})_{d\in D}$ is a profile of Nash payoff functions and for this profile of state-contingent payoffs, we can always construct (via implicit measurable selection) a corresponding profile of (measurable) functions $a(\cdot) := (a_d(\cdot))_{d\in D}$ such that given value function profile v,

$$\begin{gathered}(a_d(\omega))_{d\in D} \in \mathcal{N}(\omega, v) \text{ for all } \omega \\ \text{and} \\ (U_{\omega d})_{d\in D} = (U_d(\omega, a(\omega), v_d)_{d\in D} \text{ for all } \omega.\end{gathered}$$

Thus, given any state ω and any value function profile, we can construct, for any profile of state-contingent Nash payoffs, $(U_{(\cdot)d})_{d\in D}$, a corresponding profile of Nash equilibrium action choice functions, $(a_d(\cdot))_{d\in D}$. Therefore, if we can identify the correct one-shot game to play - or equivalently - if we can find the correct profile of value functions, then we can find a corresponding profile of Nash equilibrium action choice functions - and by Blackwell's Theorem (1965), this profile of action choice functions will form a stationary Markov equilibrium strategy for the discounted stochastic game to which our parameterized collection of one-shot games belongs.

Now we have two key observations: (1) Letting $\Sigma(\mathcal{P}(\cdot, v))$ denote the collection of all Nash payoff selection functions given $v \in \mathcal{L}^\infty_X$ (i.e., the collection of all measurable selections, $(U_{(\cdot)d})_{d\in D}$, from $\mathcal{P}(\cdot, v)$ given v), we have

$$\Sigma(\mathcal{P}(\cdot, v)) \subseteq \mathcal{L}^\infty_X \text{ for all } v \in \mathcal{L}^\infty_X$$

(i.e., all Nash payoff selection profiles are contained in the space of value function profiles, $\mathcal{L}^\infty_X$), and (2) equipping $\mathcal{L}^\infty_X$ with the appropriate topology (which in this case is the weak star or w^*-topology), we see that in light of observation (1) if we could find a fixed point of the Nash payoff selection mapping,

$$v \longrightarrow \Sigma(\mathcal{P}(\cdot, v)),$$

or equivalently, if we could find a value function profile, v^*, such that $v^*(\omega) \in \mathcal{P}(\omega, v^*)$ for all ω, then by Blackwell's Theorem applied to stochastic games, such a fixed point, say $v^* \in \Sigma(\mathcal{P}(\cdot, v^*))$, would identify the "correct" one-shot game to play. But therein lies the problem. The Nash payoff selection mapping is very badly behaved. In particular, it is, in general, neither convex-valued nor closed-valued with respect to the required w^*-topology.[6] The fact that $\Sigma(\mathcal{P}(\cdot, v))$ lacks convexity is a direct consequence of the fact that, except in very special case, the underlying Nash mapping, $\mathcal{N}(\cdot, \cdot)$, is not convex valued.[7] The fact that $\Sigma(\mathcal{P}(\cdot, v))$ is not w^*-closed valued is direct consequence of the fact that w^*-convergence does not in general imply pointwise convergence (i.e., if v^n w^*-converges to v^*, then this does not necessarily imply that $v^n(\cdot)$ converges to $v^*(\cdot)$ pointwise in R^m).[8]

These facts about the Nash payoff selection mapping under the weak star topology brings us immediately to the usefulness of introducing a public randomization device. In particular, as shown by Nowak and Raghavan (1992) and by Duffie, Geanakoplos, Mas-Colell, and McLennan (1994), we can easily extend our discounted stochastic game to include a public randomization device by simply considering instead the *convexified* Nash payoff selection mapping given by

$$v \longrightarrow \Sigma(co\mathcal{P}(\cdot, v)),$$

where "*co*" denotes the "convex hull of". Then, as shown by Nowak and Raghavan, the convexified Nash payoff selection mapping is w^*-upper semicontinuous with nonempty, convex, and w^*-compact values, and thus by the Kakutani-Glicksberg Fixed Point Theorem (Glicksberg, 1952), there exists a fixed point in valuation profiles, $v^* \in \Sigma(co\mathcal{P}(\cdot, v^*))$.[9]

[6]For example, lack of convexity alone immediately rules out the application of the Kakutani-Fan-Glicksberg Fixed Point Theorem (e.g., Glicksberg, 1952) to show that the Nash payoff selection correspondence has fixed points in stationary strategies - and hence seems to rule out a fixed point argument showing the existence of stationary solutions to players' Bellman equations for the game. Curtat (1996) and Horst (2005) rely on the Brouwer-Schauder-Tychonoff Theorem to find fixed point in a class of Lipschitz continuous strategies. This requires that they show that under their stronger assumptions the underlying one-shot game has a unique Nash equilibrium for each value function profile - hence the reason for their much stronger modeling assumptions - it forces the Nash payoff selection correspondence to be single-valued and therefore closed and convex-valued. Amir (for example Amir, 1996) instead relies on Tarski's Theorem for his fixed point establishing an equilibrium - hence the reason for Amir's weaker modeling assumptions. Here, we will establish *two new fixed point results* to show existence.

[7]This is because, in many games there are multiple Nash equilibria and only in very special cases is the set of equilibria convex. A quick example illustrates the problem. Consider the two player game,

$$(X_d, U_d(\cdot))_{d\in\{1,2\}}$$

with $X_1 = X_2 = [0, 1]$ and for $d = 1, 2$

$$U_d(x_1, x_2) = x_1x_2 - (0.1)x_d^2.$$

This game has only two Nash equilibria: $(0, 0)$ and $(1, 1)$ - clearly not forming a convex set.

[8]Consider the measure space $(\Omega, \mathcal{L}, \mu)$ where $\Omega := [0, 1]$ and μ is Lebesgue measure on $[0, 1]$, and define the following intervals:

$$I_n := \bigcup_{i=0,2,4,6,\ldots}[\tfrac{i}{2^n}, \tfrac{(i+1)}{2^n})$$

$$I'_n := [0, 1]\backslash I_n.$$

Each subinterval has the same length, namely, $\mu([\frac{i}{2^n}, \frac{(i+1)}{2^n})) = \frac{1}{2^n}$. Now consider the sequence of functions $f_n(\cdot)$ given by

$$f_n(y) := \frac{3}{2}\mathcal{I}_{I_n}(y) + \frac{1}{2}(1 - \mathcal{I}_{I_n}(y))$$

where

$$\mathcal{I}_{I_n}(y) = \begin{cases} 1 & \text{if } y \in I_n := \bigcup_{i=0,2,4,6,\ldots}[\frac{i}{2^n}, \frac{(i+1)}{2^n}) \\ 0 & \text{otherwise.} \end{cases}$$

This sequence of functions w^*-converges to the function $f^*(y) = 1$ for all $y \in [0, 1]$ - a function not equal to the pointwise limit of the sequence, $\{f_n(\cdot)\}_n$.

[9]The reason the convexified Nash payoff selection mapping is w^*-upper semicontinuous with nonempty, convex, and w^*-compact values is because, while w^*-convergence of a subsequence, $\{v^{n_k}\}_k$, of value func-

With this fixed point result in hand, Nowak and Raghavan show - as do Duffie, Geanakoplos, Mas-Colell, and McLennan - that there exists a *correlated* stationary Markov equilibrium, specified by a convexifying, public randomization device over a set of $m+1$ stationary Markov player strategy m-tuples.

1.3 Summary of Our Results

Given the description of the problem above, we see that the fundamental problem blocking the proof of existence of stationary Markov equilibria in uncountable-compact discounted stochastic games is a fixed point problem (see Theorem 1, our variation on Blackwell's Theorem) - caused by the lack of convexity and weak-star closedness (w^*-closedness) of the set of measurable selections from the Nash payoff correspondence for the underlying one-shot game. Here we introduce a two new approaches to the problem based on continuous approximation.

(1) First Approach: Approximable Nash Payoff Selection Correspondences - K-Class DSGs
Denoting by

$$\nu \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v) := \left\{U_{(\cdot)} \in \mathcal{L}_X^{\infty} : U_\omega \in \mathcal{P}(\omega, v) \text{ a.e. } [\mu]\right\}$$

the Nash payoff selection correspondence, we show that if $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a K-correspondence, then it is a w^*-w^*-USCO taking contractible values. In fact, here will show that if $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a K-correspondence (i.e., $Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ contains its K-limits) and if the dominating probability measure, μ, is nonatomic, then $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a w^*-w^*-USCO taking contractible values.[10] It then follows from results due to Gorniewicz, Granas, and Kryszewski (1991) that $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is w^*-w^*-approximable - and hence has fixed points. Thus, we show that if $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a K-correspondence, then there exists a value function profile, $v^* \in \mathcal{L}_X^{\infty}$, such that

$$v^* \in \mathcal{S}^{\infty}(\mathcal{P}_{v^*}).$$

Also, we will also show that if the DSG to which $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ belongs is noisy or is $\mathcal{G}$-nonatomic, then $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is automatically a K-correspondence.

(2) *Second Approach: Approximable Nash (Equilibrium) Correspondences - Approximable DSGs*
Denoting the Nash (equilibrium) correspondence by $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$, with USCO part

$$\mathcal{N}^{USCO} := \{\mathcal{N}(\omega, \cdot) : \omega \in \Omega\}$$

we show that if for each ω, $\mathcal{N}^{USCO}$ contains a w^*-A-USCO, $\eta(\omega, \cdot)$, taking contractible values, then $\mathcal{N}(\cdot, \cdot)$ is approximable, implying that there exists a value function profile, v^*, such that

$$v^*(\omega) \in \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu].$$

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tion profiles does not imply pointwise converge of the subsequence, it does imply pointwise convergence of some sequence of convex combinations of value function profiles in the subsequence - and this pointwise convergence of a subsequence of convex combinations of value function profiles together with the convexity of the of the mapping,

$$\omega \longrightarrow co\mathcal{P}(\omega, v)$$

make it possible to show that any weak* limit v^* (as the pointwise limit of some sequence of convex combinations) is such that

$$v^*(\omega) \in co\mathcal{P}(\omega, v) \text{ for all } \omega.$$

[10] Using methods introduced in Nowak (2003), this conclusion can be shown to hold on any probability space - nonatomic or not.

Part I

Discounted Stochastic Games

A non-cooperative m-player, non-zero sum discounted stochastic game (DSG) is given by the following primitives:

$$\left\{ \underbrace{(\Omega, B_\Omega, \mu)}_{\text{probability space of states}}, \underbrace{(\Phi_d(\omega), U_d(\omega, \cdot, v_d))_{d \in D}}_{\text{one-shot game, } \mathcal{G}(\omega,v)}, \underbrace{q(\cdot|\cdot,\cdot)}_{\text{law of motion}} \right\}, \tag{1}$$

where player $d's$ one-shot payoff function is given by

$$U_d(\omega, a_d, a_{-d}, v_d) := (1-\beta_d) r_d(\omega, a_d, a_{-d}) + \beta_d \int_\Omega v_d(\omega') q(\omega'|\omega, a_d, a_{-d}).$$

2 Primitives and Assumptions

The DSG satisfies the following list of assumptions, labeled [DSG-1]:

(1) $D =$ the set of players, consisting of m players indexed by $d = 1, 2, \ldots, m$ and each having discount rate given by $\beta_d \in (0,1)$;
(2) $(\Omega, B_\Omega, \mu) =$ the state space where Ω is a complete separable metric spaces with metric ρ_Ω, equipped with the Borel σ-field, B_Ω, upon which is defined a probability measure, μ;
(3) $X := X_1 \times \cdots \times X_m := \prod_d X_d \subset R^m$, the space of player payoff profiles, $U := (U_1, \ldots, U_m)$, such that for each player d, $X_d := [-M, M]$ and is equipped with the absolute value metric, $\rho_{X_d}(U_d, U_d') := |U_d - U_d'|$ and $X := \prod_d X_d$, is equipped with the sum metric, $\rho_X := \sum_d \rho_{X_d}$;
(4) $A := A_1 \times \cdots \times A_m := \prod_d A_d \subset Y := \prod_d Y_d$, the space of player action profiles, $a := (a_1, \ldots, a_m)$, such that for each player d, A_d with typical element a_d is a convex, compact metrizable subset of a locally convex Hausdorff topological vector space Y_d and is equipped with a metric, ρ_{A_d}, compatible with the locally convex topology inherited from Y_d, and A is equipped with the sum metric, $\rho_A := \sum_d \rho_{A_d}$;
(5) $\mathcal{L}^\infty_{X_d}$, the Banach space of all μ-equivalence classes of measurable (value) functions, $v_d(\cdot)$, defined on Ω with values in X_d a.e. $[\mu]$, equipped with metric $\rho_{w_d^*}$ compatible with the weak star topology inherited from $\mathcal{L}^\infty_R$;
(6) $\mathcal{L}^\infty_X := \mathcal{L}^\infty_{X_1} \times \cdots \times \mathcal{L}^\infty_{X_m} := \prod_d \mathcal{L}^\infty_{X_d} \subset \mathcal{L}^\infty_{R^m}$, the Banach space of all μ-equivalence classes of measurable (value) function profiles, $v(\cdot) := (v_1(\cdot), \ldots, v_m(\cdot))$, defined on Ω with values in X a.e. $[\mu]$, equipped with the sum metric $\rho_{w^*} := \sum_d \rho_{w_d^*}$ compatible with the weak star product topology inherited from

$$\mathcal{L}^\infty_{R^m} := \underbrace{\mathcal{L}^\infty_R \times \cdots \times \mathcal{L}^\infty_R}_{\text{m times}};$$

(7) $\omega \longrightarrow \Phi_d(\omega) \subset A_d$, is player $d's$ measurable action constraint correspondence, defined on Ω taking nonempty ρ_{A_d}-closed (and hence compact), convex values in A_d;

(8) $\omega \longrightarrow \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega) := \prod_d \Phi_d(\omega) \subset A$, players' measurable action constraint correspondence, defined on Ω taking nonempty ρ_A-closed (and hence compact), convex values in A;
(9) $r_d(\cdot,\cdot) : \Omega \times A \longrightarrow X$ is player $d's$ Caratheodory payoff function (i.e., $r_d(\omega,\cdot)$ is ρ_A-continuous on A and $r_d(\cdot,a)$ is (B_Ω, B_X)-measurable on X for each a);
(10) $q(\cdot|\cdot,\cdot) : \Omega \times A \longrightarrow \Delta(\Omega)$ is the law of motion defined on $\Omega \times A$ taking values in the space of probability measures on Ω, having the following properties: (i) $q(\cdot|\omega,a) << \mu$ for all $(\omega,a) \in \Omega \times A$ (i.e., each probability measure, $q(\cdot|\omega,a)$, is absolutely continuous with respect to the probability measure, μ), (ii) for each $E \in B_\Omega$, $q(E|\cdot,\cdot)$ is measurable on $\Omega \times A$, and (iii) the collection of probability density functions,

$$H_\mu := \{h(\cdot|\omega,a) : (\omega,a) \in \Omega \times A\},$$

of $q(\cdot|\omega,a)$ with respect to μ is such that for each state ω, the function

$$(a_d, a_{-d}) \longrightarrow h(\omega'|\omega, a_d, a_{-d})$$

is continuous in a and affine in a_d a.e. $[\mu]$ in ω'.

3 The Parameterized Collection of One-Shot Games

We know from Blackwell's Theorem (1965) - extended to stochastic games - that in order to find conditions sufficient to guarantee the existence of stationary Markov equilibrium, we must focus on the discounted stochastic game's underlying collection of one-shot games. This collection of one-shot games is parameterized by states and value function profiles. Thus, each value function profile, v, identifies a particular collection of state-contingent, one shot v-games. The crux of the problem is to identify the correct collection of state-contingent v-games for players to play - or more specifically to identify the correct value function profile, say v^*. This problem is a fixed point problem. Our main contribution, therefore, will take the form of a fixed point result for the nonconvex, measurable-selection-valued Nash payoff selection correspondence. Thus, as a consequence of Blackwell's Theorem, our objective will to identify conditions sufficient to guarantee that the Nash payoff selection correspondence, induced from the Nash payoff correspondence, has fixed points. We will then be able to deduce, via our fixed point results, that a correct collection of state-contingent, one-shot v-games exists, and via Blackwell's Theorem, we will be able to conclude that the discounted stochastic game to which this correct collection of state-contingent v-games belongs has a stationary Markov equilibrium.

We begin by discussing a $DSG's$ underlying parameterized collection of one-shot games.

3.1 One-Shot Games

Given discounted stochastic game,

$$DSG := \left\{(\Omega, B_\Omega, \mu), (A_d, \Phi_d(\cdot), r_d(\cdot,\cdot), \beta_d)_{d\in D}, q(\cdot|\cdot,\cdot)\right\},$$

with dominating probability measure, μ, and discount rate profile, $\beta := (\beta_1, \ldots, \beta_m)$, we have for each state-value function profile, $(\omega, v) \in \Omega \times \mathcal{L}^\infty_X$, a one-shot game given by

$$\mathcal{G}(\omega, v) := \{\Phi_d(\omega), U_d(\omega, (\cdot,\cdot), v_d)\}_{d\in D},$$

where for each action choice profile, $a = (a_d, a_{-d}) \in A$, player $d's$ expected one-shot payoff is

$$U_d(\omega, (a_d, a_{-d}), v_d)$$

$$:= (1 - \beta_d) r_d(\omega, (a_d, a_{-d})) + \beta_d \int_\Omega v_d(\omega') q(d\omega' | \omega, (a_d, a_{-d}))$$

$$= (1 - \beta_d) r_d(\omega, (a_d, a_{-d})) + \beta_d \int_\Omega v_d(\omega') h(\omega' | \omega, (a_d, a_{-d})) d\mu(\omega').$$

Letting

$$U(\omega, a, v) := (U_1(\omega, a, v_1), \ldots, U_m(\omega, a, v_m),$$

under assumptions [DSG-1], we have that in each state, $\omega \in \Omega$,

$$(a, v) \longrightarrow U(\omega, a, v) \in X,$$

is $\rho_{A \times w^*}$-continuous in $(a, v) \in A \times \mathcal{L}_X^\infty$ (see the Appendix 1: Mathematical Preliminaries).

A profile of action choices, $a^* \in \Phi(\omega)$, is a Nash equilibrium for the one-shot game, $\mathcal{G}(\omega, v)$, if for each player d

$$U_d(\omega, (a_d^*, a_{-d}^*), v_d) = \max_{a_d \in \Phi_d(\omega)} U_d(\omega, (a_d, a_{-d}^*), v_d).$$

Under assumptions [DSG-1] the one-shot game, $\mathcal{G}(\omega, v)$, always has a nonempty, ρ_A-compact set of Nash equilibria, $\mathcal{N}(\omega, v)$, and using Berge's Maximum Theorem it is straightforward to show that the *Nash correspondence*,

$$\mathcal{N}(\cdot, \cdot) : \Omega \times \mathcal{L}_X^\infty \longrightarrow P_f(A)$$

is upper Caratheodory (i.e., $\mathcal{N}(\cdot, \cdot)$ is product measurable in ω and ν and w^*-A-upper semicontinuous in v with nonempty, ρ_A-compact values). Moreover, it is straightforward to show that the *Nash payoff correspondence*,

$$\mathcal{P}(\cdot, \cdot) : \Omega \times \mathcal{L}_X^\infty \longrightarrow P_f(X),$$

given by

$$\mathcal{P}(\omega, v) := \{U \in X : U = U(\omega, a, v) \text{ for some } a \in \mathcal{N}(\omega, v)\},$$

is also upper Caratheodory (i.e., $\mathcal{P}(\cdot, \cdot)$ is product measurable in ω and ν and w^*-X-upper semicontinuous in v with nonempty, ρ_X-compact values). Note that the Nash payoff correspondence, $\mathcal{P}(\cdot, \cdot)$, is the composition of the Caratheodory player payoff functions, $U(\cdot, \cdot, \cdot)$, with the upper Caratheodory Nash correspondence, $\mathcal{N}(\cdot, \cdot)$. Sometimes we will write $\mathcal{P}(\omega, v)$ as $U(\omega, \mathcal{N}(\omega, v), v)$.

We will denote by

$$\mathcal{UC}_{w^*\text{-}A} := \mathcal{UC}(\Omega \times \mathcal{L}_X^\infty, P_f(A)) \text{ and } \mathcal{UC}_{w^*\text{-}X} := \mathcal{UC}(\Omega \times \mathcal{L}_X^\infty, P_f(X))$$

the collection of all upper Caratheodory correspondences defined on $\Omega \times \mathcal{L}_X^\infty$ taking values in $P_f(A)$ and $P_f(X)$ respectively. Thus, under assumptions [DSG-1],

$$\mathcal{N}(\cdot, \cdot) \in \mathcal{UC}_{w^*\text{-}A} \text{ and } \mathcal{P}(\cdot, \cdot) \in \mathcal{UC}_{w^*\text{-}X}.$$

3.2 From Action Choices to Strategies

Given a value function profile, $v \in \mathcal{L}_X^\infty$, the collection of one-shot games becomes a collection of state-contingent one-shot games,

$$\omega \longrightarrow \mathcal{G}(\omega, v) := \{\Phi_d(\omega), U_d(\omega, (\cdot, \cdot), v_d)\}_{d \in D},$$

with state contingent Nash correspondence, $\omega \longrightarrow \mathcal{N}(\omega, v)$, and state-contingent Nash payoff correspondence, $\omega \longrightarrow \mathcal{P}(\omega, v)$. If we were to measurably string together, state-by-state, Nash equilibria from each ω-game for a given value function profile v, we would obtain a profile of Nash equilibrium strategies,

$$\omega \longrightarrow a^*(\omega) := (a_1^*(\omega), \ldots, a_m^*(\omega)).$$

In particular, given v, we would have for each ω a Nash equilibrium, $a^*(\omega)$, for the one-shot v-game, $\mathcal{G}(\omega, v)$, in state ω. For each player d, the (B_Ω, B_{A_d})-measurable function, $a_d(\cdot) : \Omega \longrightarrow A_d$, is player $d's$ Nash equilibrium action choice strategy for the collection of one-shot state-contingent, v-games, $\mathcal{G}(\omega, v)_{\omega \in \Omega}$. We will denote this fact by writing

$$a^*(\cdot) \in \Sigma(\mathcal{N}(\cdot, v)) := \Sigma(\mathcal{N}_v).$$

Note that $a^*(\cdot) \in \Sigma(\mathcal{N}_v)$ is an everywhere (B_Ω, B_A)-measurable selection of the v-Nash correspondence, $\omega \longrightarrow \mathcal{N}(\omega, v)$.

3.3 Everywhere Nash Payoff Selections

Let $\Sigma(\mathcal{P}(\cdot, v)) := \Sigma(\mathcal{P}_v)$ denote the collection of all (B_Ω, B_X)-measurable selections of the Nash payoff correspondence, $\omega \longrightarrow \mathcal{P}(\omega, v)$. Thus, $U(\cdot) \in \Sigma(\mathcal{P}_v)$ if and only if $U(\omega) \in \mathcal{P}(\omega, v)$ for all ω. By the Measurable Implicit Function Theorem (Himmelberg, 1975, Theorem 7.1), for $U(\cdot) \in \Sigma(\mathcal{P}_v)$, there exists $a^*(\cdot) \in \Sigma(\mathcal{N}_v)$ such that $U(\omega) = U(\omega, a^*(\omega), v)$ for all ω and $U(\cdot, a^*(\cdot), v) \in \Sigma(\mathcal{P}_v)$. Conversely, if $a^*(\cdot) \in \Sigma(\mathcal{N}_v)$, then $U(\cdot, a^*(\cdot), v) \in \Sigma(\mathcal{P}_v)$.

3.4 Payoffs and Probabilities under Stationary Markov Strategies

A stationary Markov strategy for player d, is a (B_Ω, B_{A_d})-measurable function, $a_d(\cdot) : \Omega \longrightarrow A_d$, such that $a_d(\omega) \in \Phi_d(\omega)$ for all ω. Thus, the collection of all player d stationary Markov strategies is given by $\Sigma(\Phi_d)$, the collection of all (everywhere) measurable selections of $\Phi_d(\cdot)$.[11] A Markov strategy profile is given by,

$$(a_1(\cdot), \ldots, a_m(\cdot)) \in \Sigma(\Phi),$$

where

$$\Sigma(\Phi) := \prod_{d \in D} \Sigma(\Phi_d)$$

is the collection of all such profiles.

Let

$$r_d^n(a(\cdot))(\omega) := \begin{cases} r_d(\omega, a(\omega)) & \text{for } n = 1 \\ \\ \int_\Omega r_d(\omega', a(\omega'))q^{n-1}(\omega'|\omega, a(\omega)) & \text{for } n \geq 2, \end{cases} \tag{2}$$

denote the n^{th} period *expected* payoff to player d under Markov strategy profile $a(\cdot)$ starting at state ω given law of motion $q(\cdot|\cdot, \cdot)$. Here, for $n \geq 2$, $q^n(\cdot|\omega, a(\omega))$ is defined recursively by

$$\left.\begin{aligned} &q^n(E|\omega, a(\omega)) \\ &= \int_\Omega q(E|\omega', a(\omega'))q^{n-1}(\omega'|\omega, a(\omega)). \end{aligned}\right\} \tag{3}$$

[11] Thus, $a_d(\cdot) \in \Sigma(\Phi_d)$ if and only if

$$a_d(\cdot) : \Omega \longrightarrow A$$

is (B_Ω, B_{A_d})-measurable and $a_d(\omega) \in \Phi_d(\omega)$ for all ω. Such a strategy is stationary because it does not depend on time (the same strategy applies at all time points). Such a strategy is Markov because the action choice specified by the strategy in a function of the current state - and nothing else.

The discounted expected payoff to player d, with discount rate $\beta_d \in [0, 1)$, over an infinite time horizon under Markov strategy profile $a(\cdot)$ starting at state ω is given by

$$Er_d^{\infty}(a(\cdot))(\omega) := \sum_{n=1}^{\infty} \beta_d^{n-1} Er_d^n(a(\cdot))(\omega). \tag{4}$$

A stationary Markov strategy profile $a^*(\cdot) \in \Sigma(\Phi)$ is a *stationary Markov equilibrium* if for all players d and in all states ω,

$$Er_d^{\infty}(a_d^*(\cdot), a_{-d}^*(\cdot))(\omega) \geq Er_d^{\infty}(a_d'(\cdot), a_{-d}^*(\cdot))(\omega),$$

for all other strategies, $a_d'(\cdot) \in \Sigma(\Phi_d)$.

3.5 Other Continuity Properties

In the underlying one-shot game, each player's expected payoff function, $U_d(\cdot, \cdot, \cdot)$ for $d = 1, 2, \ldots, m$, is given by,

$$U_d(\omega, a, v_d) := \int_{\Omega} \underbrace{[(1 - \beta_d) r_d(\omega, a) + \beta_d v_d(\omega') h(\omega'|\omega, a)]}_{u_d(\omega, a, v_d(\omega'))} d\mu(\omega'). \tag{5}$$

Let

$$U(\omega, a, v) := (U_1(\omega, a, v_1), \ldots, U_m(\omega, a, v_m)$$

and

$$u(\omega, a, v(\omega')) := (u_1(\omega, a, v_1(\omega')), \ldots, u_m(\omega, a, v_m(\omega')).$$

(1) By part (iii) of assumption (6) we have via Scheffee's Theorem (see Billingsley, 1986, Theorem 16.11) that

$$\left.\begin{array}{l} \sup_{E \in \mathcal{B}(\Omega)} |q(E|\omega, a^n) - q(E|\omega, a^*)|_R \\ \leq \int_{\Omega} |h(\omega'|\omega, a^n) - h(\omega'|\omega, a^*)|_R \, d\mu(\omega') \longrightarrow 0, \end{array}\right\} \tag{6}$$

for any sequence of action profiles $\{a^n\}_n$ in $\Phi(\omega)$ converging to $a^* \in \Phi(\omega)$. Thus, $a^n \underset{\rho_A}{\longrightarrow} a^*$ implies that

$$\sup_{E \in \mathcal{B}(\Omega)} |q(E|\omega, a^n) - q(E|\omega, a^*)|_R \longrightarrow 0,$$

sometimes written $\|q(\cdot|\omega, a^n)) - q(\cdot|\omega, a^*)\|_{\infty} \longrightarrow 0$.

(2) As noted above, under assumptions (5) and (6), in each state, $\omega \in \Omega$, each player's expected payoff function, $(a, v_d) \longrightarrow U_d(\omega, a, v_d) \in X_d$, is $\rho_{A \times w_d^*}$-continuous in $(a, v_d) \in A \times \mathcal{L}_{X_d}^{\infty}$ - so that in each state, $\omega \in \Omega$, the X-valued function,

$$(a, v) \longrightarrow U(\omega, a, v) \in X,$$

is $\rho_{A \times w^*}$-continuous in $(a, v) \in A \times \mathcal{L}_X^{\infty}$. In fact, we can say more about the collection of functions, $U(\omega, \cdot, v) : A \longrightarrow X$, for $(\omega, v) \in \Omega \times \mathcal{L}_X^{\infty}$. In particular, as has been shown by Salon (1998) that *for each state* $\omega \in \Omega$ *the collection of functions,*

$$\{U(\omega, \cdot, v) : v \in \mathcal{L}_X^{\infty}\},$$

is uniformly equicontinuous on $\Phi(\omega)$.[12] To see this, let

$$U_{\omega v_d}(\cdot) := (1 - \beta_d) r_d(\omega, \cdot) + \beta_d \int_{\Omega} v_d(\omega') h(\omega'|\omega, \cdot)) d\mu(\omega').$$

[12]The collection, $\{U(\omega, \cdot, v) : v \in \mathcal{L}_X^{\infty}\}$, is uniformly equicontinuous if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any a and a' in $\Phi(\omega)$ with $\rho_A(a, a') < \delta$,

$$\rho_X(U(\omega, a, v), U(\omega, a', v)) < \varepsilon,$$

for all $v \in \mathcal{L}_X^{\infty}$.

For fixed ω, we have for each $v \in \mathcal{L}_X^\infty$

$$|U_{\omega v_d}(a) - U_{\omega v_d}(a')|$$
$$\leq (1-\beta_d)\,|r_d(\omega,a) - r_d(\omega,a')|$$
$$+\beta_d M \left| \int_\Omega h(\omega'|\omega,a)d\mu(\omega') - \int_\Omega h(\omega'|\omega,a')d\mu(\omega') \right|.$$

Because

$$r_d(\omega,\cdot) \text{ and } H_\omega(a) := \int_\Omega h(\omega'|\omega,\cdot)d\mu(\omega')$$

are continuous functions on a compact set, and hence uniformly continuous, for any $\frac{\varepsilon}{2} > 0$ there is a $\delta > 0$ such that for any a and a' in $\Phi(\omega)$ with $d_A(a,a') < \delta$

$$|r_d(\omega,a) - r_d(\omega,a'))| < \tfrac{\varepsilon}{2}$$
$$\text{and}$$
$$|H_\omega(a) - H_\omega(a')| < \tfrac{\varepsilon}{2}.$$

3.6 Nash Payoff Selections

3.6.1 The Definition

A Nash payoff selection is a function, $U_{(\cdot)} \in \mathcal{L}_X^\infty$ such that $U_\omega \in \mathcal{P}(\omega,v)$ a.e. $[\mu]$ for some fixed value function profile, $v \in \mathcal{L}_X^\infty$. Given parameterized games, $\{\mathcal{G}(\omega,v)\}_{(\omega,v)\in\Omega\times\mathcal{L}_X^\infty}$, satisfying assumptions [DSG-1] with Nash payoff correspondence, $\mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{w^*\text{-}X}$, the induced *Nash payoff selection correspondence* is given by

$$v \longrightarrow S^\infty(\mathcal{P}_v) := \left\{ U_{(\cdot)} \in \mathcal{L}_X^\infty : U_\omega \in \mathcal{P}(\omega,v) \text{ a.e. } [\mu] \right\}.$$

Thus, for each value function profile v, $\mathcal{S}^\infty(\mathcal{P}_v)$ is the set of all μ-equivalence classes of measurable selections of the measurable correspondence,

$$\omega \longrightarrow \mathcal{P}_v(\omega) := \mathcal{P}(\omega,v),$$

Recall that $\Sigma^\infty(\mathcal{P}_v)$ denotes the prequotient of $S^\infty(\mathcal{P}_v)$ while $\Sigma(\mathcal{P}_v)$ denotes the set of all everywhere measurable selections of $\mathcal{P}_v(\cdot)$ (for a given v).[13] Because the Nash payoff correspondence, $\omega \longrightarrow \mathcal{P}_v(\omega)$, is (B_Ω, B_X)-measurable with nonempty compact values in X, by the Kuratowski-Ryll-Nardzewski Selection Theorem (1965), $\mathcal{P}_v(\cdot)$ has (B_Ω, B_X)-measurable selections (i.e., $S^\infty(\mathcal{P}_v) \neq \varnothing$).

3.6.2 Decomposability

In general, a subset $\mathcal{S}$ of $\mathcal{L}_X^\infty$ is said to be decomposable if for any two functions $U^0_{(\cdot)}$ and $U^1_{(\cdot)}$ in $\mathcal{S}$ and for any $E \in B_\Omega$, we have

$$U^0_{(\cdot)} I_E(\cdot) + U^1_{(\cdot)} I_{\Omega\backslash E}(\cdot) \in \mathcal{S}.$$

For the Nash payoff correspondence, $\mathcal{P}(\cdot,\cdot) : \Omega \times \mathcal{L}_X^\infty \longrightarrow P_f(X)$, an upper Caratheodory correspondence, for each $v \in \mathcal{L}_X^\infty$, the induced Nash payoff selection correspondence, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, takes decomposable values. Moreover, for each v, $\mathcal{S}^\infty(\Gamma_v)$ is $\|\cdot\|_1$-closed (or $\mathcal{L}^1_{R^m}$-closed) in $\mathcal{L}^\infty_{R^m}$. Thus, for any sequence $\{U^n_{(\cdot)}\}_n$ in $\mathcal{S}^\infty(\mathcal{P}_v)$ converging in $\mathcal{L}^1_{R^m}$-norm

[13] A (B_Ω, B_X)-measurable function, $U_{(\cdot)}$, is an everywhere measurable selection of $\mathcal{P}_v(\cdot)$ provided $U_\omega \in \mathcal{P}_v(\omega)$ for all $\omega \in \Omega$.

to $U^0_{(\cdot)} \in \mathcal{L}^\infty_{R^m}$, we have $U^0_{(\cdot)} \in \mathcal{S}^\infty(\mathcal{P}_v)$. We will denote by $cl_1\mathcal{S}^\infty(\mathcal{P}_v)$ the $\mathcal{L}^1_{R^m}$-closure of $\mathcal{S}^\infty(\mathcal{P}_v)$ in $\mathcal{L}^\infty_{R^m}$. By Lemma 1 in Pales and Zeidan (1999), we know that, in addition to $\mathcal{S}^\infty(\mathcal{P}_v)$ being decomposable, $\mathcal{S}^\infty(\mathcal{P}_v)$ is $\mathcal{L}^1_{R^m}$-closed in $\mathcal{L}^\infty_{R^m}$. Thus, we have

$$cl_1\mathcal{S}^\infty(\mathcal{P}_v) = \mathcal{S}^\infty(\mathcal{P}_v).$$

We also know by Corollary 1 in Pales and Zeidan (1999) that

$$cl_1\mathcal{S}^\infty(\mathcal{P}_v) = \left\{U_{(\cdot)} \in \mathcal{L}^\infty_{R^m} : \exists \, \{U^n_{(\cdot)}\}_n \ \subset \mathcal{S}^\infty(\mathcal{P}_v) \text{ such that } \lim_n \left\| U^n_{(\cdot)} - U_{(\cdot)} \right\|_1 = 0 \right\}.$$

Finally, note that $\mathcal{L}^\infty_X$ too is decomposable and $\mathcal{L}^1_{R^m}$-closed in $\mathcal{L}^\infty_{R^m}$.

3.6.3 Sequences of Nash Payoff Selections and Sequences of Nash Equilibria

Consider a sequence,

$$\left\{(v^n, U^n_{(\cdot)}\right\}_n \subset GrS^\infty(\mathcal{P}_{(\cdot)}) \subset \mathcal{L}^\infty_X \times \mathcal{L}^\infty_X,$$

where for each n, $U^n_{(\cdot)} \in \mathcal{L}^\infty_X$ is a Nash payoff selection, that is,

$$U^n_\omega \in \mathcal{P}(\omega, v^n) \text{ a.e. } [\mu].$$

Let N^∞ be the exceptional set (i.e., the set of μ-measure zero) such for $\omega \in \Omega \backslash N^\infty$, $U^n_\omega \in \mathcal{P}(\omega, v^n)$ for all n.

For each n, we have by the Measurable Implicit Function Theorem (e.g., Himmelberg, 1975, Theorem 7.1) a (B_Ω, B_A)-measurable function, $a^n(\cdot) : \Omega \longrightarrow A$, such that for each n and $\omega \in \Omega \backslash N^\infty$, $a^n(\omega) \in \mathcal{N}(\omega, v^n)$. Thus, we have for each n and $\omega \in \Omega \backslash N^\infty$,

$$U^n_\omega = U(\omega, a^n(\omega), v^n) \text{ for all } \omega \in \Omega \backslash N^\infty, \tag{7}$$

where

$$U(\omega, a^n(\omega), v^n) := (U_1(\omega, a^n(\omega), v^n_1), \ldots, U_m(\omega, a^n(\omega), v^n_m)).$$

Note that under assumptions [DSG-1], the sequence $\{U^n_{(\cdot)}\}_n \subset \mathcal{L}^1_{R^m}$ is $\|\cdot\|_1$-bounded.

3.7 Selections from the Graph of the Nash Payoff Correspondence

3.7.1 The Definition

A measurable selection from the graph of the Nash payoff correspondence,

$$\omega \longrightarrow Gr\mathcal{P}_\omega(\cdot) := \left\{(v, U) \in \mathcal{L}^\infty_X \times X : U \in \mathcal{P}(\omega, v)\right\},$$

is a function, $\omega \longrightarrow (v_\omega, U_\omega) \in \mathcal{L}^\infty_X \times X$, such that $U_\omega \in \mathcal{P}(\omega, v_\omega)$ for all ω. By Lemma 3.1(ii) in Kucia and Nowak (2000), the Nash payoff graph correspondence, $Gr\mathcal{P}_{(\cdot)}(\cdot)$, is $(B_\Omega, B_{w^*} \times B_X)$-measurable with nonempty compact values in $\mathcal{L}^\infty_X \times X$, and by the Kuratowski-Ryll-Nardzewski Selection Theorem (1965), the Nash payoff graph correspondence, $Gr\mathcal{P}_{(\cdot)}(\cdot)$, has $(B_\Omega, B_{w^*} \times B_X)$-measurable selections (i.e., $\Sigma(Gr\mathcal{P}_{(\cdot)}(\cdot)) \neq \varnothing$, so that $\mathcal{S}^\infty(Gr\mathcal{P}_{(\cdot)}(\cdot)) \neq \varnothing$). Thus, there exists at least one measurable function,

$$\omega \longrightarrow (v_\omega, U_\omega) \in \mathcal{L}^\infty_X \times X$$

such that

$$U_\omega \in \mathcal{P}(\omega, v_\omega) \text{ for all } \omega.$$ [14]

3.7.2 Sequences of Payoff Graph Selections and Sequences of Nash Equilibria

Consider a sequence, $\left\{(v^n_{(\cdot)}, U^n_{(\cdot)}\right\}_n \subset \Sigma(Gr\mathcal{P}_{(\cdot)}(\cdot))$, of selections of the Nash payoff graph correspondence. For each n,

$$U^n_\omega \in \mathcal{P}(\omega, v^n_\omega) \text{ for all } \omega,$$

and therefore, for each n, we have by the Measurable Implicit Function Theorem (e.g., Himmelberg, 1975, Theorem 7.1) a (B_Ω, B_A)-measurable function, $a^n(\cdot) : \Omega \longrightarrow A$, such that for each n, $a^n(\omega) \in \mathcal{N}(\omega, v^n_\omega)$ for all ω with

$$U^n_\omega = U(\omega, a^n(\omega), v^n_\omega) \text{ for all } \omega, \tag{8}$$

where

$$U(\omega, a^n(\omega), v^n_\omega) := (U_1(\omega, a^n(\omega), v^n_{\omega 1}), \ldots, U_m(\omega, a^n(\omega), v^n_{\omega m})).$$

3.8 The Problem

It follows from Blackwell's Theorem (1965) that a stationary Markov strategy profile,

$$a^*(\cdot) := (a^*_1(\cdot), \ldots, a^*_m(\cdot)) \in \Sigma(\mathcal{N}_{v^*}),$$

is a Nash equilibrium of a discounted stochastic game if and only if there exists a profile of continuation values (or value functions), $v^* := (v^*_1, \ldots, v^*_m) \in \mathcal{L}^\infty_X$ such that $v^*(\omega) \in \mathcal{P}(\omega, v^*)$ for all ω, i.e., such that,

$$v^*(\cdot) := (v^*_1(\cdot), \ldots, v^*_m(\cdot)) \in \Sigma(\mathcal{P}_{v^*}),$$

and such that together the pair, $(a^*(\cdot), v^*(\cdot)) \in \Sigma(\mathcal{N}_{v^*}) \times \Sigma(\mathcal{P}_{v^*})$. Equivalently, $a^*(\cdot)$ is a stationary Markov equilibrium if and only if the pair, $(a^*(\cdot), v^*(\cdot))$, satisfy the following system of equations:

for players $d = 1, 2, \ldots, m$ and for all initial states ω

$$v^*_d(\omega) = \underbrace{(1-\beta_d)r_d(\omega, a^*(\omega)) + \beta_d \int_\Omega v^*_d(\omega')h(\omega'|\omega, a^*(\omega))d\mu(\omega')}_{U_d(\omega, a^*(\omega), v^*_d)} \tag{9}$$

and

$$U_d(\omega, a^*_d(\omega), a^*_{-d}(\omega), v^*_d) = \max_{a \in \Phi_d(\omega)} U_d(\omega, a, a^*_{-d}(\omega), v^*_d). \tag{10}$$

Thus, if for the given strategy profile, $a^*(\cdot)$, $v^*(\cdot)$, satisfies state-by-state for each player d the Bellman equations (9), and if for the given value function profile, $v^*(\cdot)$, $a^*(\cdot)$, satisfies state-by-state for each player d the Nash conditions (10), then together, $(a^*(\cdot), v^*(\cdot))$, satisfy Blackwell's conditions, and by Blackwell's Theorem, $a^*(\cdot)$ is a stationary Markov equilibrium of the discounted stochastic game with underlying state-contingent, collection of one-shot games, $\{\mathcal{G}(\omega, v^*)\}_{\omega \in \Omega}$.

Note that if $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$ is the Nash equilibria correspondence for the one-shot game, $(\omega, v) \longrightarrow \mathcal{G}(\omega, v)$, and if $(\omega, v) \longrightarrow \mathcal{P}(\omega, v)$ is the induced Nash equilibria payoff correspondence given by

$$\mathcal{P}(\omega, v) := \{U \in X : U_d = U_d(\omega, a, v_d) \forall d \text{ and some } a \in \mathcal{N}(\omega, v)\}$$

then by Blackwell's Theorem (1965) the discounted stochastic game with underlying collection of one-shot games,

$$\mathcal{G}(\omega, v)_{(\omega, v) \in \Omega \times \mathcal{L}^\infty_X},$$

has a stationary Markov equilibrium if and only if there is a value function profile, $\overline{v}^*$, such that

$$\overline{v}^*(\omega) \in \mathcal{P}(\omega, \overline{v}^*) \text{ a.e. } [\mu],$$

or equivalently, if and only if there is a value function profile, $\overline{v}^*$, such that

$$\overline{v}^* \in \mathcal{S}^\infty(\mathcal{P}(\cdot, \overline{v}^*)),$$

where for each v, $\mathcal{S}^\infty(\mathcal{P}(\cdot, v))$ is the set of μ-equivalence classes of measurable selections of the Nash payoff correspondence, $\omega \longrightarrow \mathcal{P}(\omega, v)$. Once we have found a fixed point,

$$\overline{v}^* \in \mathcal{S}^\infty(\mathcal{P}_{\overline{v}^*}) := \mathcal{S}^\infty(\mathcal{P}(\cdot, \overline{v}^*)),$$

or equivalently a solution to the Bellman inclusion and in particular, a $\overline{v}^* \in \mathcal{L}^\infty_X$ such that

$$\overline{v}^*(\omega) \in \mathcal{P}(\omega, \overline{v}^*) \text{ a.e. } [\mu],$$

we can easily deduce the existence of an everywhere measurable selection $v^* \in \Sigma(\mathcal{P}(\cdot, v^*))$ such that $v^* = \overline{v}^*$ a.e. $[\mu]$ and from this we can easily deduce the existence of the strategy profile, $a^*(\cdot)$, such that $a^*(\cdot) \in \Sigma(\mathcal{N}(\cdot, v^*))$ using the Measurable Implicit Function Theorem (e.g., Himmelberg, 1975, Theorem 7.1). Thus, in order to establish the existence of a stationary Markov equilibrium for our discounted stochastic game it follows from Blackwell's Theorem (1965) that it is both necessary and sufficient that there exists a fixed point, v^*, of the corresponding the Nash payoff selection correspondence, $v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v))$ or equivalently, that the Bellman inclusion have a solution. Formally, we have the following variation on Blackwell's Theorem (1965):

Theorem 1 *(Necessary and sufficient conditions for the existence of stationary Markov equilibria):*
Let

$$DSG := \left\{(\Omega, B_\Omega, \mu), (A_d, \Phi_d(\cdot), r_d(\cdot,\cdot), \beta_d)_{d \in D}, q(\cdot|\cdot,\cdot)\right\},$$

be a discounted stochastic game satisfying assumptions [DSG-1], with Nash payoff correspondence, $\mathcal{P}(\cdot,\cdot)$, for the underlying one-shot game. Then DSG has a stationary Markov equilibrium if and only if the Nash payoff selection correspondence,

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v)),$$

has a fixed point.

Part II
The Fixed Point Problem for Nash Payoff Selection Correspondences

As a consequence of Theorem 1 above, the first solution we will present for the problem of existence of stationary Markov equilibria in discounted stochastic games will take the form of a new fixed point theorem for the nonconvex, measurable-selection-valued Nash payoff selection correspondence.

Consider the measurable-selection-valued correspondence,

$$\nu \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}(\cdot, v)) := \left\{ U_{(\cdot)} \in \mathcal{L}_X^{\infty} : U_{\omega} \in \mathcal{P}(\omega, v) \text{ a.e. } [\mu] \right\},$$

induced by an upper Caratheodory correspondence,

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := \{ U \in X : U = U(\omega, a, v) \text{ for some } a \in \mathcal{N}(\omega, v) \},$$

gotten by composing the continuous function, $U(\omega, \cdot, v) : A \longrightarrow X$, with the upper Caratheodory correspondence, $\mathcal{N}(\cdot, \cdot) : \Omega \times \mathcal{L}_X^{\infty} \longrightarrow P_{\rho_A f}(A)$. Here, $P_{\rho_A f}(A)$ is the collection of all nonempty, ρ_A-closed subsets of A. We will sometimes denote this composition by

$$(\omega, v) \longrightarrow U(\omega, \mathcal{N}(\omega, v), v).$$

We will use the notations $v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}(\cdot, v))$, $v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v)$, and $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ to denote our measurable-selection-valued correspondence. In general, the induced measurable selection valued correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, of an upper Caratheodory correspondence, $\mathcal{P}(\cdot, \cdot)$, while nonempty valued is neither convex-valued nor closed-valued in the weak star topology - and these facts make the fixed point problem for such correspondences difficult.

4 Approximable Nash Payoff Selection Correspondences

Definition 1 *(Approximable Nash Payoff Selection Correspondences)*
$\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is w^*-w^*-approximable if for each $\varepsilon > 0$, there exists a w^*-w^*-continuous function,

$$g^{\varepsilon}(\cdot) : \mathcal{L}_X^{\infty} \longrightarrow \mathcal{L}_X^{\infty},$$

such that for each $(v, U_{(\cdot)}) \in Grg^{\varepsilon} \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$ (i.e., for each $(v, U_{(\cdot)}) \in \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$, with $U_{(\cdot)} = g^{\varepsilon}(v) \in \mathcal{L}_X^{\infty}$) there exists

$$(\overline{v}, \overline{U}_{(\cdot)}) \in Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$$

(i.e., there exists $\overline{U}_{(\cdot)} \in \mathcal{S}^{\infty}(\mathcal{P}_{\overline{v}})$) such that

$$\rho_{w^*}(v, \overline{v}) + \rho_{w^*}(U_{(\cdot)}, \overline{U}_{(\cdot)}) < \varepsilon. \tag{11}$$

Equivalently, for any $\varepsilon > 0$

$$Grg^{\varepsilon} \subset B_{w^* \times w^*}(\varepsilon, Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})).$$

Thus, the graph of the continuous function $g^{\varepsilon} : \mathcal{L}_X^{\infty} \longrightarrow \mathcal{L}_X^{\infty}$ is contained in the $\rho_{w^* \times w^*}$-open ball of radius ε about the graph of $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$.

5 DSGs with Nash Payoff Selection Correspondences that are K-Correspondence

We begin by defining what we mean by a K-correspondence.

Definition 2 *(K Correspondences)*
Consider the Nash payoff correspondence given by

$$\mathcal{P}(\omega, v) := U(\omega, \mathcal{N}(\omega, v), v),$$

where the upper Caratheodory Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, and the X-valued Caratheodory players' payoff function, $U(\cdot,\cdot,\cdot)$, satisfy the relevant assumptions in list, [DSG-1]. Consider the induced Nash payoff selection correspondence,

$$v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v).$$

We say that $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a K correspondence (or has the K-limit Property) if for any K-converging sequence,

$$\{(v^n, U^n_{(\cdot)})\}_n \subset Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) \subset \mathcal{L}^{\infty}_X \times \mathcal{L}^{\infty}_X,$$

with K-limit $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^{\infty}_X \times \mathcal{L}^{\infty}_X$,

$$\widehat{U}_{(\cdot)} \in \mathcal{S}^{\infty}(\mathcal{P}_{\widehat{v}}).$$

By Page's (1991) lower closure result for K-limits (Proposition 1 in Page, 1991), we know that

$$\widehat{U}_{\omega} \in coLs\{U^n_{\omega}\} \text{ a.e. } [\mu]. \tag{12}$$

This fact will be very useful in connecting our results on stationary Markov equilibria to recent results by Duggan (2012) and He and Sun (2015).

5.1 $\mathcal{G}$-Nonatomic Discounted Stochastic Games

In this subsection we will define the notion of $\mathcal{G}$-nonatomic DSGs and we will show that all $\mathcal{G}$-nonatomic DSGs have Nash payoff selection correspondences that are K-correspondences. This conclusion rests upon a measure theoretic condition introduced by Rokhlin (1949) and Dynkin and Evstigneev (1976) ensuring the existence of a convex set of conditional selections of a measurable, closed valued correspondence. We will call this condition the $\mathcal{G}$-nonatomic condition.

The $\mathcal{G}$-nonatomic condition led He and Sun (2015) to study the class of DSGs which we will call here, the $\mathcal{G}$-nonatomic class. In this subsection, using (12) and Dynkin and Evstigneev (1976) we will show that all $\mathcal{G}$-nonatomic DSGs are K-class (i.e., $\mathcal{G}$-nonatomic DSGs are a sub-class of DSGs with the K-limit property). He and Sun (2013) call the class of $\mathcal{G}$-nonatomic DSGs, *games with a coarser transition kernels*. Whatever its name, the usefulness of the $\mathcal{G}$-nonatomic class in establishing the existence of stationary Markov equilibria follows from the extension of Lyapunov's Theorem (1940) due to Dynkin and Evstigneev (1976). In what we do here, we go back to the definitions and results of Dynkin and Evstigneev (1976) - rather than He and Sun (2015). Recall that here we have assumed that the state space is a Polish space, Ω, equipped with the Borel σ-field, B_{Ω}, and a probability measure, μ, defined on B_{Ω}. Also, recall that when Ω is Polish, μ is nonatomic if and only if $\mu(\{\omega\}) = 0$ for all $\omega \in \Omega$ (see Hildenbrand, 1974). Suppose now that $\mathcal{G}$ is a sub-σ-field of B_{Ω}. Denote by $\mu^{\mathcal{G}}(\cdot)$ a regular $\mathcal{G}$-conditional probability given sub-σ-field $\mathcal{G}$. Following Dynkin and Evstigneev, $A \in B_\Omega$ is $\mathcal{G}$-atom if $\mu(A) > 0$ and for any $B \in B_\Omega$ such that $B \subset A$

$$\mu \left\{ \omega \in \Omega : 0 < \mu^{\mathcal{G}}(B)(\omega) < \mu^{\mathcal{G}}(A)(\omega) \right\} = 0.$$

Let $\Gamma : \Omega \longrightarrow P_f(X)$ be an arbitrary measurable correspondence taking nonempty, closed values in X. We will denote by

$$\mathcal{S}^\infty_{\mathcal{G}}(\Gamma) := \{E(U|\mathcal{G}) \in \mathcal{L}^\infty_X(\mathcal{G}) : U \in \mathcal{S}^\infty(\Gamma)\}$$

the collection of all μ-equivalence classes of regular $\mathcal{G}$-conditional expectations of μ-essentially bounded a.e. measurable selections of Γ. The following extension of Lyapunov's Theorem is due to Dynkin and Evstigneev (1976).

Theorem 2 *(An extension of Lyapunov's Convexity Theorem)*
Let $\Gamma : \Omega \longrightarrow P_f(X)$ be a measurable correspondence taking nonempty, closed values in X. If for some sub-σ-field, $\mathcal{G}$, of B_Ω, B_Ω contains no $\mathcal{G}$-atoms, then

$$\mathcal{S}^\infty_{\mathcal{G}}(\Gamma) = \mathcal{S}^\infty_{\mathcal{G}}(co\Gamma),$$

where co denotes the convex hull.

He and Sun (2015) give a slightly different definition of $\mathcal{G}$-atoms - one implied by Dynkin and Evstigneev's definition - and they show that if the state space underlying the game is nonatomic and has no $\mathcal{G}$-atoms, then the the discounted stochastic game has a stationary Markov equilibrium. Our next result, Theorem 3, shows that our condition, the K-limit property, is implied by the absence of $\mathcal{G}$-atoms.

Theorem 3 *(All $\mathcal{G}$-nonatomic DSGs are K-class nonatomic DSGs)*
Let $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^\infty_X}$ be the parameterized one-shot game underlying a discounted stochastic game, DSG, satisfying assumptions [DSG-1] and having a Nash payoff selection correspondence, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$. If the underlying probability space, (Ω, B_Ω, μ), is such that for some sub-σ-field, $\mathcal{G}$, of B_Ω, B_Ω contains no $\mathcal{G}$-atoms, then $\mathcal{S}^\infty_{\mathcal{G}}(\mathcal{P}_{(\cdot)})$ has the K-limit property, and in fact, is a convex-valued, w^*-w^*-sub-USCO of $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$.

PROOF: Let $\{(v^n, U^n_{(\cdot)})\}_n$ be any sequence contained in $Gr(\mathcal{S}^\infty(\mathcal{P}_{(\cdot)}))$ such that $v^n \underset{w^*}{\longrightarrow} v^* \in \mathcal{L}^\infty_X$. We have for each n, $U^n(\omega) \in \mathcal{P}(\omega, v^n)$ a.e. $[\mu]$. By the K compactness of $\mathcal{L}^\infty_X$, we can assume WLOG that the sequence, $\{(v^n, U^n_{(\cdot)})\}_n$, K converges with K limit $(\widehat{v}, \widehat{U}_{(\cdot)}) \in \mathcal{L}^\infty_X \times \mathcal{L}^\infty_X$. We have

$$\widehat{U}^n(\omega) := \frac{1}{n}\sum_{k=1}^{n} U^k(\omega) \longrightarrow \widehat{U}(\omega) \text{ a.e. } [\mu],$$

and by the properties of conditional expectations (see Ash, 1972),

$$E(\widehat{U}^n|\mathcal{G})(\omega) := \frac{1}{n}\sum_{k=1}^{n} E(U^k|\mathcal{G})(\omega) \longrightarrow E(\widehat{U}|\mathcal{G})(\omega) \text{ a.e. } [\mu].$$

By Proposition 1 in Page (1991) - i.e., by Page's lower closure result,

$$E(\widehat{U}|\mathcal{G})(\omega) \in coLs\{E(U^n|\mathcal{G})(\omega)\} \text{ a.e. } [\mu].$$

By Dynkin and Evstigneev (1976),

$$coLs\{E(U^n|\mathcal{G})(\omega)\} = Ls\{E(U^n|\mathcal{G})(\omega)\} \text{ a.e. } [\mu].$$

and by the properties of conditional expectations, $coLs\{E(U^n|\mathcal{G})(\cdot)\} \subset \mathcal{S}^{\infty}_{\mathcal{G}}(\mathcal{P}_{v^*})$. Thus,

$$E(\widehat{U}|\mathcal{G})(\omega) \in Ls\{E(U^n|\mathcal{G})(\omega)\} \text{ a.e. } [\mu],$$

i.e., $\mathcal{S}^{\infty}_{\mathcal{G}}(\mathcal{P}_{(\cdot)})$ has the K-limit property. In fact, $\mathcal{S}^{\infty}_{\mathcal{G}}(\mathcal{P}_{(\cdot)})$ is a convex-valued, and by Theorem A2.3 is a w^*-w^*-sub-USCO of $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$. Q.E.D.

5.2 Noisy Discounted Stochastic Games

Another interesting sub-class of discounted stochastic games is the class of noisy DSGs recently studied by Duggan (2012). By specializing primitives and assumptions of our discounted stochastic game model above, we can easily make our model a noisy stochastic game model. We need only modify assumptions (2) and (10) as follows:

In a noisy DSG (i.e., NDSG) the state space is given by $\Omega := T \times S$ with typical element $t := (t, s)$, where both T and S are complete separable metric spaces with metrics ρ_T and ρ_S, equipped with the Borel σ-fields B_T and B_S. By structuring the state space in this way, we can analyze situations where part of the riskiness is controllable (in a stochastic sense) and part of the riskiness is only indirectly controllable or not controllable at all. In particular, we can think of $t \in T$ as being the stochastically controllable regular state, and we can think of $s \in S$ as being the indirectly stochastically controllable (or uncontrollable) noisy state.

In an NDSG, the law of motion

$$(\underbrace{(t, s)}_{\omega}, a) \longrightarrow q(\cdot|\underbrace{(t, s)}_{\omega}, a)$$

is given by

$$q(d(t', s')|(t, s), a) := \varepsilon(ds'|t')\delta(dt'|(t, s), a),$$

or

$$q(d(t', s')|\omega, a) := \varepsilon(ds'|t')\delta(dt'|\omega, a),$$

where $\omega = (t, s)$ denotes the current state and $\omega' = (t', s')$ denotes the coming state - and depending on the regular state t' chosen by the probability measure, $\delta(dt'|\omega, a)$, in current state $\omega = (t, s)$ given action profile $a \in \Phi(\omega)$, the noisy state s' will be chosen according to the probability measure, $\varepsilon(ds'|t')$. Thus, while regular states are directly stochastically controllable via the stochastic kernel, $\delta(dt'|\omega, a)$, noisy states are only indirectly stochastically controllable via $\varepsilon(ds'|t')$. In this sense, we say that the discounted stochastic game is noisy.

To complete our formal description of the noisy discounted stochastic game model, assume that for all $t' \in T$, the probability measure, $\varepsilon(ds'|t')$, governing the choice of the coming noisy state s' is absolutely continuous with respect to a probability measure, $\lambda(ds')$, defined on the measurable space, (S, B_S), of noisy states.[15] Also, assume that for all $(\omega, a) \in \Omega \times \Phi(\omega)$, the probability measure, $\delta(dt'|\omega, a)$, governing the choice of the coming regular state t', given current state, $\omega := (t, s)$ and action profile $a \in \Phi(\omega)$, is absolutely continuous with respect to a probability measure, $\gamma(dt')$, defined on the measurable space, (T, B_T), of regular states. Thus, the noisy DSG has dominating probability

[15] Duggan assumes that the dominating probability measure, λ, is nonatomic - but we will show that this is not required for existence of stationary Markov equilibria.

measure given by the product measure, $\mu := \lambda \times \gamma$. By the Corollary in Rao and Rao (1972), if λ is nonatomic, then μ is nonatomic.[16]

Let

$$G_\gamma := \{g(dt|\omega, a) : (\omega, a) \in \Omega \times A\},$$

be the collection of probability density functions of $\delta(\cdot|\omega, a)$ with respect to γ such that for each state $\omega := (t, s)$, the function

$$(a_d, a_{-d}) \longrightarrow g(t'|(t, s), a_d, a_{-d})$$

is continuous in a and affine in a_d a.e. $[\gamma]$ in t'. Also, let

$$K_\lambda := \{r(ds'|t') : t' \in T\},$$

be the collection of probability density functions of $\varepsilon(\cdot|t')$ with respect to λ such that the function

$$t' \longrightarrow r(s'|t')$$

is measurable in t' a.e. $[\lambda]$ in s'.

Specializing (2) and (10) in our list of assumptions, [DSG-1], above, label the new list of assumptions [NSG-1].

The key result connecting Duggan's noisy DSGs to our K-class DSGs is due to He and Sun (2015).

Theorem 4 *(All Noisy DSGs Are $\mathcal{G}$-Nonatomic)*
If $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}_X^\infty}$ is the parameterized one-shot game underlying a discounted stochastic game, DSG, satisfying assumptions [NSG-1](i.e., if DSG is noisy), then DSG is $\mathcal{G}$-nonatomic.

Thus, by Theorems 3 and 4 we have,

$$\{\text{all noisy } DSGs\} \subset \{\text{all } \mathcal{G}\text{-nonatomic } DSGs\} \subset \{\text{all } K\text{-class } DSGs\}.$$

He and Sun (2013) were the first to investigate $\mathcal{G}$-nonatomic discounted stochastic games - only they used a weaker version of the Dynkin-Evstigneev-Rokhlin condition. They called their condition - implied by the Dynkin-Evstigneev-Rokhlin condition - the coarser transition kernel condition. Both $\mathcal{G}$-nonatomic discounted stochastic games, as well as discounted stochastic games with coarser transition kernel are examples of K-class discounted stochastic games. The key ingredient allowing the $\mathcal{G}$-nonatomic condition as well as the coarser transition kernel condition to deliver an existence result for stationary Markov equilibria is the extension of Lyapunov's Theorem (1940) due to Dynkin and Evstigneev (1976). In what we did here, we went back to the definitions and results of Dynkin and Evstigneev (1976) - rather than He and Sun (2015).[17]

[16] $E \subset S$ is an atom of S relative to $\lambda(\cdot)$ if the following implication holds: if $\lambda(E) > 0$, then $H \subset E$ implies that $\lambda(H) = 0$ or $\lambda(E - H) = 0$. If S contains no atoms relative to $\lambda(\cdot)$, S is said to be atomless or nonatomic. Because S, is a complete, separable metric space $\lambda(\cdot)$ is atomless (or nonatomic) if and only if $\lambda(\{\omega\}) = 0$ for all $\omega \in S$ (see Hildenbrand, 1974, pp 44-45).

[17] He and Sun (2015) give a slightly different definition of $\mathcal{G}$-atoms - one implied by Dynkin and Evstigneev's definition - and they show that if the state space underlying the game is nonatomic and has no $\mathcal{G}$-atoms, then the the discounted stochastic game has a stationary Markov equilibrium.

6 K-Class DSGs and the Fixed Point Problem

Recall that the Nash payoff selection correspondence,

$$v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}(\cdot,v)) := \mathcal{S}^{\infty}(U(\cdot,\mathcal{N}(\cdot,v),v)),$$

for the parameterized game,

$$\mathcal{G}(\omega,v)_{(\omega,v)\in\Omega\times\mathcal{L}_X^{\infty}} := \left(\{\Phi_d(\omega), U_d(\omega,(\cdot,\cdot),v_d)\}_{d\in D}\right)_{(\omega,v)\in\Omega\times\mathcal{L}_X^{\infty}},$$

underlying a discounted stochastic game - in most cases - has very undesirable properties. However, in this subsection, we will show that, in fact, if the Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, is a K-correspondence, then $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is an approximable, w^*-w^*-USCO - and therefore, a correspondence having fixed points.

6.1 All Nonatomic K-Class DSGs have Approximable Nash Payoff Selection Correspondences

We will show that if the Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, belonging to a DSG is a K-correspondence, then it is a w^*-w-USCO. Moreover, we show that if the dominating probability measure, μ, is nonatomic, then $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ takes contractible values (with respect to the w^* topology). Thus, if the DSG is noisy and the dominating probability measure for the noisy state, λ, is nonatomic, then $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a w^*-w^*-USCO taking contractible values.

Theorem 5 *(If $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ has the K-limit property and μ is nonatomic, then $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a w^*-w^*-USCO taking contractible values)*
Let $\{\mathcal{G}(\omega,v)\}_{(\omega,v)\in\Omega\times\mathcal{L}_X^{\infty}}$ be the parameterized one-shot game underlying a discounted stochastic game satisfying assumptions [DSG-1]. If the Nash payoff selection correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, is a K-correspondence (i.e., has the K-limit property), then the following statements are true:
(1) $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is a w^*-w^*-USCO, that is, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) \in \mathcal{U}_{w^*\text{-}w^*}$.
(2) If the dominating probability measure, μ, is nonatomic, then for each $v \in \mathcal{L}_X^{\infty}$, $\mathcal{S}^{\infty}(\mathcal{P}_v)$ is contractible (with respect to the w^* topology).

PROOF: (1) Because $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ has the K-limit property, it follows from Komlos Theorem and Theorem A2.1(1) that for each $v \in \mathcal{L}_X^{\infty}$, $\mathcal{S}^{\infty}(\mathcal{P}_v)$ is w^*-compact. Therefore, to show that $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) \in \mathcal{U}_{w^*\text{-}w^*}$, it suffices to show that $Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is $\rho_{w^*\times w^*}$-closed in $\mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$. Let $\{(v^n, U^n_{(\cdot)})\}_n$ be any sequence in $Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ such that

$$v^n \underset{K}{\longrightarrow} \widehat{v} \text{ and } U^n_{(\cdot)} \underset{K}{\longrightarrow} \widehat{U}_{(\cdot)}.$$

Thus, $\{(v^n, U^n_{(\cdot)})\}_n$ is a sequence of payoff selections (rather than a sequence of payoff graph selections). By Theorem A2.1(1), we have $v^n \underset{w^*}{\longrightarrow} v^*$ and $U^n_{(\cdot)} \underset{w^*}{\longrightarrow} U^*_{(\cdot)}$ with $v^* = \widehat{v}$ and $\widehat{U}_{(\cdot)} = U^*_{(\cdot)}$ a.e. $[\mu]$. Also, we have for each n,

$$U^n_{\omega} \in \mathcal{P}(\omega, v^n) \text{ a.e. } [\mu].$$

By the K-limit property of $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, we have that

$$\widehat{U}_{(\cdot)} \in \mathcal{S}^{\infty}(\mathcal{P}_{\widehat{v}}),$$

and given that $\widehat{U}_{(\cdot)} = U^*_{(\cdot)}$ a.e. $[\mu]$, we have

$$U^*_{(\cdot)} \in \mathcal{S}^\infty(\mathcal{P}_{\widehat{v}}).$$

Given that $v^* = \widehat{v}$ a.e. $[\mu]$, we have $\mathcal{P}(\omega, v^*) = \mathcal{P}(\omega, \widehat{v})$ a.e. $[\mu]$. Thus, $U^*_{(\cdot)} \in \mathcal{S}^\infty(\mathcal{P}_{v^*})$ (i.e., implying that $(v^*, U^*_{(\cdot)}) \in Gr\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$.

(2) Next, for $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)}) \in \mathcal{U}_{w^*\text{-}w^*}$, we will show that if the dominating probability measure, μ, is nonatomic, then for each v, $\mathcal{S}^\infty(\mathcal{P}_v)$ is contractible.

First, if the dominating probability measure, μ, is nonatomic, then as shown by Fryszkowski (1983), Liapunov's Theorem (1940) on the range of a vector measure guarantees the existence of a family of measurable sets, $\{E_t\}_{t\in[0,1]}$, such that

$$\left.\begin{array}{c} t' \leq t \Rightarrow E_{t'} \subseteq E_t,\ E_0 = \varnothing \text{ and } E_1 = \Omega, \text{ and} \\ \mu(E_t) = t\mu(\Omega) = t. \end{array}\right\} \tag{13}$$

Using the properties of this system of measurable sets and the decomposability of $\mathcal{S}^\infty(\mathcal{P}_v)$ for each $v \in \mathcal{L}^\infty_X$, we will show that for each v the function $h_v(\cdot,\cdot)$ given by

$$h_v(U,t) := U^1_{(\cdot)} I_{E_t}(\cdot) + U_{(\cdot)} I_{\Omega\setminus E_t}(\cdot) \in s(v) \text{ for all } (U,t) \in \mathcal{S}^\infty(\mathcal{P}_v) \times [0,1] \tag{14}$$

is a homotopy (and in particular, a contraction of $\mathcal{S}^\infty(\mathcal{P}_v)$ to U^1). Here $v \in \mathcal{L}^\infty_X$ is fixed, $I_E(\cdot)$ is the indicator function of set E, and $U^1_{(\cdot)}$ is any fixed selection in $\mathcal{S}^\infty(\mathcal{P}_v)$.

It suffices to show that $h_v(\cdot,\cdot)$ is $\rho_{w^*\times|\cdot|}$-ρ_{w^*}-continuous. Let $\{(U^n_{(\cdot)}, t^n)\}_n$ be such a sequence such that

$$U^n_{(\cdot)} \underset{w^*}{\longrightarrow} U^*_{(\cdot)} \text{ and } t^n \underset{R}{\longrightarrow} t^*.$$

We must show that

$$h_v(U^n_{(\cdot)}, t^n) \underset{w^*}{\longrightarrow} h_v(U^*_{(\cdot)}, t^*) \in s(v).$$

It suffices to show that for all $l \in \mathcal{L}^1_{R^m}$ with $\|l\|_1 \leq 1$,

$$H = \textstyle\int_\Omega \left\langle \left(U^1_\omega I_{E_{t^n}}(\omega) - U^1_\omega I_{E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)$$

$$+ \textstyle\int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega) \longrightarrow 0.$$

Rewriting, expression H, we have

$$H = \underbrace{\int_\Omega \left\langle \left(U^1_\omega I_{E_{t^n}}(\omega) - U^1_\omega I_{E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(a)}$$

$$+ \underbrace{\int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^n}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)}_{(b)}$$

$$+ \underbrace{\int_\Omega \left\langle \left(U^*_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega).}_{(c)}$$

Because $U^n \underset{w^*}{\longrightarrow} U^*$, we have

$$\textstyle\int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^n}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)$$

$$+ \textstyle\int_\Omega \left\langle \left(U^n_\omega I_{E_{t^n}}(\omega) - U^*_\omega I_{E_{t^n}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)$$

$$= \textstyle\int_\Omega \left\langle \left(U^n_\omega - U^*_\omega\right), l(\omega)\right\rangle d\mu(\omega) \longrightarrow 0.$$

Thus, $(b) \longrightarrow 0$. Given that $X_d = [-M, M]$ for all d, we note that each of the expressions (a) and (c) is less than or equal to $2M \|l\|_1 \mu(E_{t^n} \vartriangle E_{t^*})$, and given that $\|l\|_1 \leq 1$, we have $(a) + (c) \leq 4M\mu(E_{t^n} \vartriangle E_{t^*})$. We have, then,

$$\int_\Omega \left\langle \left(U^1_\omega I_{E_{t^n}}(\omega) - U^1_\omega I_{E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)$$

$$+ \int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^*}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega)$$

$$\leq 4M\mu(E_{t^n} \vartriangle E_{t^*}) + \int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^n}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega),$$

and as n goes to infinity

$$4M\mu(E_{t^n} \vartriangle E_{t^*})$$

$$+ \int_\Omega \left\langle \left(U^n_\omega I_{\Omega\setminus E_{t^n}}(\omega) - U^*_\omega I_{\Omega\setminus E_{t^n}}(\omega)\right), l(\omega)\right\rangle d\mu(\omega) \longrightarrow 0.$$

Thus, the $\rho_{w^* \times |\cdot|}$-$\rho_{w^*}$-continuous function given in (14) for each $v \in \mathcal{L}^\infty_X$, together with the properties of the Liapunov system (13) specify a homotopy for the set of measurable selections, $\mathcal{S}^\infty(\mathcal{P}_v)$ - and thus for each v, $\mathcal{S}^\infty(\mathcal{P}_v)$ is contractible. Q.E.D.

Our proof that $\mathcal{S}^\infty(\mathcal{P}_v)$ is contractible for each v is a modified version of the proof given by Mariconda (1992) showing that if the underlying probability space is nonatomic then any decomposable subset of E-valued, Bochner integrable functions in $\mathcal{L}^1_E$ is contractible (where E is a Banach space). In Mariconda's result, the space of functions is equipped with the norm in $\mathcal{L}^1_E$, while here our space of functions (with each function taking values in $X \subset R^m$) is equipped with the metric, ρ_{w^*}, a metric compatible with the w^* topology.

We close this subsection with two results: First, we show that because the Nash payoff selection correspondence, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, is a contractibly-valued (a consequence of $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ being a K-correspondence and the probability space being nonatomic), the w^*-w^*-USCO, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, is w^*-w^*-approximable. Second, we show that because $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, is w^*-w^*-approximable, it has fixed points.

Theorem 6 *(If $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is a K correspondence and μ nonatomic, then $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is w^*-w^*-approximable)*
Let $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^\infty_X}$ be the parameterized one-shot game underlying a discounted stochastic game satisfying assumptions [DSG-1] with Nash payoff selection correspondence $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$. If $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is a K-correspondence, and if the dominating probability measure, μ, is nonatomic, then $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is a w^*-w^*-approximable.

PROOF: By Corollary 5.6 in Gorniewicz, Granas, and Kryszewski (1991), because the sub-USCO, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, is defined on the ANR (absolute neighborhood retract) the space of value functions $\mathcal{L}^\infty_X$ taking nonempty, compact, and contractible values in $\mathcal{L}^\infty_X$ (and hence ∞-proximally connected values - see Theorem 5.3 in Gorniewicz, Granas, and Kryszewski, 1991), the w^*-w^*-USCO, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, is a J mapping. Therefore, by Theorem 5.12 in Gorniewicz, Granas, and Kryszewski (1991), $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is w^*-w^*-approximable. Q.E.D.

We can now state our main fixed point result.

Theorem 7 *(Fixed points for w^*-w^*-approximable Nash payoff selection correspondences)*
Let $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^\infty_X}$ be the parameterized one-shot game underlying a discounted stochastic game satisfying assumptions [DSG-1] with Nash payoff selection correspondence $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$. If $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is a K-correspondence, and if the dominating probability measure, μ, is nonatomic, then $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ has a fixed point (i.e., there exists $v^* \in \mathcal{L}^\infty_X$ such that $v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*})$).

PROOF: By Theorem 6 above, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is w^*-w^*-approximable. Therefore, we have for each n, a w^*-w^*-continuous function,

$$g^n(\cdot) : \mathcal{L}_X^{\infty} \longrightarrow \mathcal{L}_X^{\infty},$$

such that for each $(v^n, U_{(\cdot)}^n) \in Grg^n \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$ (i.e., for each $(v^n, U_{(\cdot)}^n) \in \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$, with $U_{(\cdot)}^n = g^n(v^n) \in \mathcal{L}_X^{\infty}$) there exists

$$(\overline{v}^n, \overline{U}_{(\cdot)}^n) \in Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}) \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$$

(i.e., there exists $\overline{U}_{(\cdot)}^n \in \; Gr\mathcal{S}^{\infty}(\mathcal{P}_{\overline{v}^n})$) such that

$$\rho_{w^*}(v^n, \overline{v}^n) + \rho_{w^*}(U_{(\cdot)}^n, \overline{U}_{(\cdot)}^n) < \frac{1}{n^2}. \tag{15}$$

Equivalently, for any positive integer, n,

$$Grg^n \subset B_{w^* \times w^*}(\tfrac{1}{n^2}, Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})).$$

Thus, the graph of the continuous function $g^n : \mathcal{L}_X^{\infty} \longrightarrow \mathcal{L}_X^{\infty}$ is contained in the $\rho_{w^* \times w^*}$-open ball of radius $\frac{1}{n^2}$ about the graph of $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$.

Because each of the functions, g^n, is w^*-w^*-continuous and defined on the w^*-compact and convex subset, $\mathcal{L}_X^{\infty}$, in $\mathcal{L}_{R^m}^{\infty}$, taking values in $\mathcal{L}_X^{\infty}$, it follows from the fixed point theorem of Schauder (see Aliprantis and Border, 2006), that each g^n has a fixed point, $v^n \in \mathcal{L}_X^{\infty}$ (i.e., for each n there exists some $v^n \in \mathcal{L}_X^{\infty}$ such that $v^n = g^n(v^n)$). Let $\{v^n\}_n$ be a fixed point sequence corresponding to the sequence of w^*-w^*-continuous approximating functions, $\{g^n(\cdot)\}_n$. Expression (15) can now be rewritten as follows: for each v^n in the fixed point sequence, there is a corresponding pair, $(\overline{v}^n, \overline{U}_{(\cdot)}^n) \in Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, such that

$$\rho_{w^*}(v^n, \overline{v}^n) + \rho_{w^*}(g^n(v^n), \overline{U}_{(\cdot)}^n) < \tfrac{1}{n^2},$$

and therefore such that

$$\underbrace{\rho_{w^*}(v^n, \overline{v}^n)}_{A} + \underbrace{\rho_{w^*}(v^n, \overline{U}_{(\cdot)}^n)}_{B} < \tfrac{1}{n^2}. \tag{16}$$

By the w^*-compactness of $\mathcal{L}_X^{\infty}$, we can assume WLOG that the fixed point sequence, $\{v^n\}_n \subset \mathcal{L}_X^{\infty}$, w^*-converges to a limit $v^* \in \mathcal{L}_X^{\infty}$. Thus, by part A of (16), as $n \longrightarrow \infty$ we have

$$v^n \underset{w^*}{\longrightarrow} v^* \text{ and } \overline{v}^n \underset{w^*}{\longrightarrow} v^*,$$

and therefore by part B of (16), as $n \longrightarrow \infty$ we have

$$\overline{U}^n \underset{w^*}{\longrightarrow} v^*.$$

Because $Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is $\rho_{w^* \times w^*}$-closed in $\mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$,

$$\{(\overline{v}^n, \overline{U}_{(\cdot)}^n)\}_n \subset Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}),$$

and $\overline{v}^n \underset{w^*}{\longrightarrow} v^*$ and $\overline{U}_{(\cdot)}^n \underset{w^*}{\longrightarrow} v^*$ imply that

$$(v^*, v^*) \in Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)}).$$

Therefore, $v^* \in \mathcal{S}^{\infty}(\mathcal{P}_{v^*})$. Q.E.D.

Given assumptions [DSG-1] it follows from Theorem 1 (Blackwell's Theorem) and Theorem 7 above that *all K-class nonatomic discounted stochastic games (including noisy DSGs and nonatomic DSGs satisfying the coarser transition kernel condition) have stationary Markov equilibria*.

If the graph of the Nash payoff selection correspondence, $Gr\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, contains a K-closed subset, $s \subset \mathcal{L}_X^{\infty} \times \mathcal{L}_X^{\infty}$ whose domain is all of $\mathcal{L}_X^{\infty}$ (i.e., $proj_{\mathcal{L}_X^{\infty}}(s) = \mathcal{L}_X^{\infty}$), then

$$v \longrightarrow s(v) := \left\{ U_{(\cdot)} \in \mathcal{L}_X^{\infty} : (v, U_{(\cdot)}) \in s \right\}$$

is a w^*-w^*-sub-USCO belonging to $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, and moreover, if the dominating probability measure, μ, is nonatomic, then $s(v) := \left\{ U_{(\cdot)} \in \mathcal{L}_X^{\infty} : (v, U_{(\cdot)}) \in s \right\}$ is contractible.

Part III

The Measurable Selection Problem for Nash Payoff Correspondences

The second solution we will present for the problem of existence of stationary Markov equilibria in discounted stochastic games will take the form of a new selection theorem for the nonconvex, R^m-valued Nash payoff correspondence. As we have seen, this correspondence is gotten by composing players' Caratheodory payoff functions with the upper Caratheodory Nash equilibrium correspondence.

The USCO part of the Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, is given by

$$\mathcal{N}^{USCO} := \{\mathcal{N}(\omega,\cdot) \in \mathcal{U}_{w^*\text{-}A} : \omega \in \Omega\}.$$

Under assumptions [A-1], we will show that if in each state ω the USCO part of $\mathcal{N}(\cdot,\cdot)$ contains an approximable sub-USCO, $\eta(\omega,\cdot)$ - for example, if for each ω the sub-USCO,

$$v \longrightarrow \eta_\omega(v) := \eta(\omega, v),$$

is R_δ-valued (for example, convex valued, or more generally, contractibly valued) - then there exists $v^* \in \mathcal{L}^\infty_X$, such that

$$v^*(\omega) = \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu],$$

implying that

$$v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*}).$$

It is interesting to note that if for each ω, $\eta_\omega(\cdot)$ is a minimal USCO belonging to $\mathcal{N}_\omega(\cdot)$, then $\eta_\omega(v)$ is single valued - and hence contractibly valued - for v is a w^*-dense subset of the parameter space, $\mathcal{L}^\infty_X$. Thus, if on the w^*-meager subset of $\mathcal{L}^\infty_X$ where $\eta_\omega(\cdot)$ is multivalued, $\eta_\omega(\cdot)$ takes connected, locally connected, and hereditarily unicoherent values, then $\eta_\omega(\cdot)$ will be contractibly valued for all $v \in \mathcal{L}^\infty_X$. Page (2013) has shown, under the same assumptions on the primitives as those made here, that if in addition, in each state ω the parameterized collection, $\{U(\omega,\cdot,v) : v \in \mathcal{L}^\infty_X\}$, is uniformly equicontinuous, then all minimal USCOs belonging to the USCO part of $\mathcal{N}(\cdot,\cdot)$ are essentially-valued (in the sense of Fort, 1950) as well as connected-valued. Thus, save for a meager set, under assumptions [A-1] and the uniform equicontinuity of

$$\{U(\omega,\cdot,v) : v \in \mathcal{L}^\infty_X\}$$

for each ω, $\mathcal{P}(\cdot,\cdot) := U(\cdot,\mathcal{N}(\cdot,\cdot),v\cdot)$ is by its very nature close to having an induced selection correspondence,

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v) := \mathcal{S}^\infty(U(\cdot,\mathcal{N}(\cdot,v),v)),$$

possessed of fixed points.

7 Approximable Nash Correspondences

Let $\mathcal{U}_{w^*\text{-}A} := \mathcal{U}(\mathcal{L}^\infty_X, P_{\rho_A f}(A))$ denote the collection of all upper semicontinuous correspondences taking nonempty, ρ_A-closed (and hence ρ_A-compact) values in A. Following the literature, we will call such mappings, USCOs (see Crannell, Franz, and LeMasurier, 2005, Anguelov and Kalenda, 2009, and Hola and Holy, 2009). Given any $\mathcal{A} \in \mathcal{U}_{w^*\text{-}A}$, denote by $\mathcal{U}_{w^*\text{-}A}[\mathcal{A}]$ the collection of all sub-USCOs belonging to $\mathcal{A}$, that is, all USCOs $\alpha \in \mathcal{U}_{w^*\text{-}A}$ whose graph,

$$Gr\alpha := \{(v, a) \in \mathcal{L}_X^\infty \times A : a \in \alpha(v)\},$$

is contained in the graph of $\mathcal{A}$,

$$Gr\mathcal{A} := \{(v, a) \in \mathcal{L}_X^\infty \times A : a \in \mathcal{A}(v)\}.$$

We will call any sub-USCO, $\alpha \in \mathcal{U}_{w^*\text{-}A}[\mathcal{A}]$ a minimal USCO belonging to $\mathcal{A}$, if for any other sub-USCO, $\beta \in \mathcal{U}_{w^*\text{-}A}[\mathcal{A}]$, $Gr\beta \subseteq Gr\alpha$ implies that $Gr\beta = Gr\alpha$ (see Drewnowski and Labuda, 1990). We will denote by $[\mathcal{A}]$ the collection of all minimal USCOs belonging to $\mathcal{A}$.

We begin with a formal definition of approximable.

Definition 3 *(Approximable Upper Caratheodory Nash Correspondences):*
We say that the upper Caratheodory Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, is approximable if the USCO part,

$$\mathcal{N}^{USCO} := \{\mathcal{N}(\omega, \cdot) \in \mathcal{U}_{w^*\text{-}A} : \omega \in \Omega\},$$

is such that in each state ω there is a sub-USCO,

$$\eta(\omega, \cdot) \in \mathcal{U}_{w^*\text{-}A}[\mathcal{N}(\omega, \cdot)],$$

such that for any $\varepsilon > 0$, there exists a w^*-A-continuous function,

$$g_\omega^\varepsilon(\cdot) : \mathcal{L}_X^\infty \longrightarrow A,$$

having the property that for each $(v, g_\omega^\varepsilon(v)) \in \mathcal{L}_X^\infty \times A$ there exists $(\overline{v}, \overline{a}) \in Gr\eta(\omega, \cdot)$ such that

$$\rho_{w^*}(v, \overline{v}) + \rho_{w^*}(g_\omega^\varepsilon(v), \overline{a}) < \varepsilon,$$

or equivalently, such that for any $\varepsilon > 0$, there exists a w^*-A-continuous function, $g_\omega^\varepsilon(\cdot) : \mathcal{L}_X^\infty \longrightarrow A$, having the property that

$$Grg_\omega^\varepsilon \subset B_{\rho_{w^* \times A}}(\varepsilon, Gr\eta(\omega, \cdot)),$$

where $B_{\rho_{w^* \times A}}(\varepsilon, Gr\eta(\omega, \cdot))$ is $\rho_{w^* \times A}$-open enlargement of $Gr\eta(\omega, \cdot)$ consisting of those points, (v, a), in $\mathcal{L}_X^\infty \times A$ at less than ε distance from $Gr\eta(\omega, \cdot)$.

A function $U^\varepsilon : \Omega \times \mathcal{L}_X^\infty \longrightarrow A$ is Caratheodory if it is (B_Ω, B_A)-measurable in ω for each v and w^*-A-continuous in v for each ω.

Definition 4 *(Caratheodory Approximable Upper Caratheodory Correspondences):*
We say that an upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, is Caratheodory approximable if for any $\varepsilon > 0$, there exists a Caratheodory function,

$$g^\varepsilon : \Omega \times \mathcal{L}_X^\infty \longrightarrow A,$$

having the property that for each $(\omega, v) \in \Omega \times \mathcal{L}_X^\infty$ and each $(v, g^\varepsilon(\omega, v)) \in \mathcal{L}_X^\infty \times A$ there exists $(\overline{v}, \overline{a}) \in Gr\mathcal{N}(\omega, \cdot)$ such that

$$\rho_{w^*}(v, \overline{v}) + \rho_A(g^\varepsilon(\omega, v), \overline{a}) < \varepsilon,$$

or equivalently, such that for any $\varepsilon > 0$, there exists a Caratheodory function, $g^\varepsilon : \Omega \times \mathcal{L}_X^\infty \longrightarrow A$, having the property that for each ω

$$Grg^\varepsilon(\omega, \cdot) \subset B_{\rho_{w^* \times A}}(\varepsilon, Gr\mathcal{N}(\omega, \cdot)).$$

The following result on Caratheodory approximable upper Caratheodory correspondences (specialized to our game-theoretic model) is due to Kucia and Nowak (2000, Theorem 4.2).

Theorem 8 *(Approximable implies Caratheodory approximable): Suppose assumptions [A-1](1)-(7) hold. If the upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, is approximable, then $\mathcal{N}(\cdot,\cdot)$ is Caratheodory approximable.*

7.1 A Selection Theorem for Approximable Nash Correspondences

In this section, we will show that if the upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, is approximable, then its induced measurable-selection-valued correspondence, $S^{\infty}(\mathcal{P}_{(\cdot)})$, has fixed points. We begin with our main selection result.

Theorem 9 *(A selection result for approximable upper Caratheodory correspondences) Suppose assumptions [DSG-1] hold and let,*

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := \{U \in X : U = U(\omega, a, v) \text{ for some } a \in \mathcal{N}(\omega, v)\},$$

be the upper Caratheodory Nash payoff correspondence. If $\mathcal{N}(\cdot,\cdot)$ is approximable, then there exists $v^* \in \mathcal{L}_X^{\infty}$ such that

$$v^*(\omega) \in \mathcal{P}(\omega, v^*) \ \text{a.e. } [\mu].$$

PROOF: Because $\mathcal{N}(\cdot,\cdot)$ is approximable, it is Caratheodory approximable. Thus, for each n, there exists a Caratheodory $\frac{1}{n^2}$-approximation,

$$g^n(\cdot,\cdot) : \Omega \times \mathcal{L}_X^{\infty} \longrightarrow A,$$

of $\mathcal{N}(\cdot,\cdot)$. Consider the sequence of functions,

$$v \longrightarrow T_v^n(\cdot) := U(\cdot, g^n(\cdot, v), v) \in \mathcal{L}_X^{\infty}. \tag{17}$$

Observe that for each n, $T_{(\cdot)}^n(\cdot)$ is a function from $\mathcal{L}_X^{\infty}$ into $\mathcal{L}_X^{\infty}$. Moreover, note that for each n the function $T_{(\cdot)}^n(\cdot)$ is w^*-w^*-continuous (i.e., $v^k \underset{\rho_{w^*}}{\longrightarrow} v^*$ implies that $T_{v^k}^n(\cdot) \underset{w^*}{\longrightarrow} T_{v^*}^n(\cdot)$). This is true because for each n, $v^k \underset{\rho_{w^*}}{\longrightarrow} v^*$ implies that for each $\omega \in \Omega$, as $k \longrightarrow \infty$, $g^n(\omega, v^k) \underset{\rho_A}{\longrightarrow} g^n(\omega, v^*) \in A$, and therefore for each $\omega \in \Omega$,

$$U(\omega, g^n(\omega, v^k), v^k) \underset{R^m}{\longrightarrow} U(\omega, g^n(\omega, v^*), v^*),$$

implying that

$$U(\cdot, g^n(\cdot, v^k), v^k) \underset{w^*}{\longrightarrow} U(\cdot, g^n(\cdot, v^*), v^*) \in \mathcal{L}_X^{\infty}.$$

By the Schauder-Tychonoff Fixed Point Theorem (e.g., see Aliprantis-Border, 2006), for each n, there exists $v^n \in \mathcal{L}_X^{\infty}$ such that

$$v^n(\cdot) = U(\cdot, g^n(\cdot, v^n), v^n). \tag{18}$$

Thus, we have for each n a set, N^n, of μ-measure zero such that

$$v^n(\omega) = U(\omega, g^n(\omega, v^n), v^n) \text{ for all } \omega \in \Omega \backslash N^n,\ \mu(N^n) = 0. \tag{19}$$

Call the equation (19), one for each n, the Caratheodory equation and call the sequence, $\{v^n\}_n$, in $\mathcal{L}_X^\infty$ the *Caratheodory fixed point sequence* and let $N^\infty := \cup_n N^n$ - so that, $\mu(N^\infty) = 0$.

For each fixed point and Caratheodory approximating function pair, $(v^n, g^n(\cdot, v^n))$, consider the measurable function,

$$\omega \longrightarrow \min_{(v,a)\in Gr\mathcal{N}_\omega(\cdot)}[\rho_{w^*}(v^n, v) + \rho_A(g^n(\omega, v^n), a)]. \tag{20}$$

The graph correspondence,

$$\omega \longrightarrow Gr\mathcal{N}_\omega(\cdot),$$

is measurable (by Kucia-Nowak, 2000) and compact-valued, and therefore, by the continuity of the function

$$(v, a) \longrightarrow [\rho_{w^*}(v^n, v) + \rho_A(g^n(\omega, v^n), a)]$$

on $\mathcal{L}_X^\infty \times A$, there exists for each n, a measurable selection of $Gr\mathcal{N}_{(\cdot)}(\cdot)$,

$$\omega \longrightarrow (v_\omega^n, a_\omega^n) \in \mathcal{L}_X^\infty \times A$$

solving the minimization problem (20) state by state (see Himmelberg, Parthasarathy, Raghavan, and Van Vleck, 1976). Thus, for the measurable function, $\omega \longrightarrow (v_\omega^n, a_\omega^n)$, we have

$$\left.\begin{array}{c} \omega \longrightarrow (v_\omega^n, a_\omega^n) \in Gr\mathcal{N}_\omega(\cdot) \text{ for all } \omega \in \Omega \\ \\ (\text{i.e., } a_\omega^n \in \mathcal{N}(\omega, v_\omega^n)\ \forall \omega \in \Omega), \end{array}\right\} \tag{21}$$

and

$$[\rho_{w^*}(v^n, v_\omega^n) + \rho_A(g^n(\omega, v^n), a_\omega^n)] = \min_{(v,a)\in Gr\mathcal{N}_\omega(\cdot)}[\rho_{w^*}(v^n, v) + \rho_A(g^n(\omega, v^n), a)],$$

so by Theorem 8 above (i.e., the Kucia-Nowak result), we know that

$$\underbrace{\rho_{w^*}(v^n, v_\omega^n)}_{A} + \underbrace{\rho_A(g^n(\omega, v^n), a_\omega^n)}_{B} < \tfrac{1}{n^2} \text{ for all } \omega \in \Omega. \tag{22}$$

Next, let $\omega \longrightarrow (v_\omega^*, a_\omega^*)$ be a measurable selection from the correspondence

$$\omega \longrightarrow Ls_{\rho_{A\times w^*}}\{(a_\omega^n, v_\omega^n)\}.$$

Because the Nash equilibrium correspondence, $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$, has a closed graph,

$$a_\omega^* \in \mathcal{N}(\omega, v_\omega^*) \text{ for all } \omega \in \Omega.$$

Because $v^n \underset{w^*}{\longrightarrow} v^*$, we have by part A of (22) that

$$v_\omega^n \underset{w^*}{\longrightarrow} v^* \in \mathcal{L}_X^\infty \text{ for all } \omega \in \Omega,$$

and by (19), part B of (22) and the continuity properties of $U(\omega, \cdot, \cdot)$ we have that,

$$v^*(\omega) = U(\omega, a_\omega^*, v^*) \text{ a.e. } [\mu].$$

where $(a_\omega^*, v^*) \in Ls_{\rho_{A\times w^*}}\{(a_\omega^n, v_\omega^n)\}$ for all ω. Finally, because

$$a_\omega^* \in \mathcal{N}(\omega, v^*) \text{ for all } \omega,$$

we have for any measurable selection, $(a^*_{(\cdot)}, v^*)$, from

$$\omega \longrightarrow Ls_{\rho_{A\times w^*}}\{(a^n_\omega, v^n_\omega)\}$$

that

$$v^*(\omega) = U(\omega, a^*_\omega, v^*) \in \mathcal{P}(\omega, v^*) \text{ for all } \omega \in \Omega\backslash N^\infty,\ \mu(N^\infty) = 0.$$

Q.E.D.

An immediate Corollary of Theorem 9 is the following fixed point result.

Corollary to Theorem 9 *(Fixed points for Nash payoff selection correspondences induced from approximable Nash correspondences)*
Suppose assumptions [DSG-1] hold and let

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := \{U \in X : U = U(\omega, a, v) \text{ for some } a \in \mathcal{N}(\omega, v)\} := U(\omega, \mathcal{N}(\omega, v), v)$$

be an upper Caratheodory Nash payoff correspondence. If the Nash correspondence, $\mathcal{N}(\cdot,\cdot)$*, is approximable, then there exists* $v^* \in \mathcal{L}^\infty_X$ *such that*

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v)$$

has fixed points (i.e., there exists $v^* \in \mathcal{L}^\infty_X$ *such that* $v^* \in \mathcal{S}^\infty(\Gamma_{v^*})$*).*

PROOF: By Theorem 9, there exists $v^* \in \mathcal{L}^\infty_X$ such that

$$v^*(\omega) \in \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu].$$

Therefore,

$$v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*}).$$

Q.E.D.

Given assumptions [DSG-1] it follows from Theorem 1 (Blackwell's Theorem) and Theorem 9 above that *all approximable discounted stochastic games have stationary Markov equilibria.*

7.2 Conditions Sufficient for a DSG to be Approximable

We will agree that a discounted stochastic game is approximable if the underlying parameterized one-shot game,

$$\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^\infty_X},$$

satisfying assumptions [DSG-1] has Nash correspondence,

$$\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{w^*\text{-}A},$$

which possesses for each ω a sub-USCO,

$$\eta(\omega,\cdot) \in \mathcal{U}_{w^*\text{-}A}[\mathcal{N}(\omega,\cdot)],$$

that is w^*-A-approximable. Recall that a sub-USCO, $\eta(\omega,\cdot)$, is w^*-A-approximable if for each $\omega \in \Omega$ and for each $\varepsilon > 0$, there exists a w^*-A-continuous function, $g^\varepsilon_\omega : \mathcal{L}^\infty_X \longrightarrow A$, such that for each $(v, g^\varepsilon_\omega(v)) \in \mathcal{L}^\infty_X \times A$ there exists $(\overline{v}, \overline{a}) \in Gr\eta(\omega,\cdot)$ such that

$$\rho_{w^*}(v, \overline{v}) + \rho_A(g^\varepsilon_\omega(v), \overline{a}) < \varepsilon.$$

Our objective in this section is to identify conditions sufficient to guarantee that the Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, has an USCO part,

$$\{\mathcal{N}_\omega(\cdot) : \omega \in \Omega\} \subset \mathcal{U}_{w^*\text{-}A},$$

such that for each ω, there exists some w^*-A-approximable sub-USCO,

$$\eta(\omega,\cdot) \in \mathcal{U}_{w^*\text{-}A}[\mathcal{N}_\omega(\cdot)].$$

Before we state our main results, recall the following facts from metric topology (see the Appendix 3 for a more complete summary): A space Z is R_δ provided there is a sequence of AR spaces, $\{Z^n\}_n$, such that $Z^{n+1} \subset Z^n$ for all n with $Z = \cap_{n=1}^{\infty} Z^n$.[18] Also recall from Gorniewicz, Granas, and Kryszewski (1991) that an USCO taking ∞-proximally connected values is called a J-mapping. For example, if $\eta(\omega,\cdot) \in \mathcal{U}_{w^*\text{-}A}[\mathcal{N}_\omega(\cdot)]$ is such that for each $v \in \mathcal{L}_X^\infty$, $\eta(\omega, v)$ is R_δ-valued (and hence ∞-proximally connected valued) then $\eta(\omega,\cdot)$, is a J-mapping.

Our main result on approximability gives conditions on the sub-USCOs, $\eta(\omega,\cdot) \in \mathcal{U}_{w^*\text{-}A}[\mathcal{N}_\omega(\cdot)]$, sufficient to guarantee w^*-A-approximability.

Theorem 10 *(Sufficient conditions for approximability)*
Let $\{\mathcal{G}(\omega, v)\}_{(\omega,v)\in\Omega\times\mathcal{L}_X^\infty}$ be the parameterized one-shot game underlying a discounted stochastic game, DSG, satisfying assumptions [DSG-1] and having a Nash correspondence,

$$\mathcal{N}(\cdot,\cdot) : \Omega \times \mathcal{L}_X^\infty \longrightarrow P_f(A),$$

with USCO part is given by

$$\mathcal{N}^{USCO} := \{\mathcal{N}(\omega,\cdot) \in \mathcal{U}_{w^*\text{-}A} : \omega \in \Omega\}.$$

If for each ω there exists a sub-USCO $\eta(\omega,\cdot) \in \mathcal{U}_{w^*\text{-}A}[\mathcal{N}_\omega(\cdot)]$ with *$v \longrightarrow \eta(\omega, v)$ taking R_δ values in $\Phi(\omega)$, then DSG is approximable.*

PROOF: Let $\eta(\omega,\cdot) \in \mathcal{U}_{w^*\text{-}A}[\mathcal{N}_\omega(\cdot)]$, and for each ω, consider the the Nash payoff sub-USCO, $\eta(\omega,\cdot)$. By Corollary 5.6 in Gorniewicz, Granas, and Kryszewski (1991), because $\eta(\omega,\cdot)$ is a mapping defined on the ANR (absolute neighborhood retract) space of value functions $\mathcal{L}_X^\infty$ taking nonempty, compact, R_δ values in the ANR space $\Phi(\omega)$, the Nash payoff sub-USCO, $\eta(\omega,\cdot)$, is ∞-proximally connected valued, and therefore a J mapping. Thus, by Theorem 5.12 in GGK (1991), $\eta(\omega,\cdot)$ is w^*-A-approximable, and therefore, the DSG is approximable. Q.E.D.

If the Nash USCO, $\mathcal{N}(\omega,\cdot)$, has at least one sub-USCO, $\eta(\omega,\cdot)$, taking R_δ values in $\Phi(\omega)$, then the set of Nash equilibria generating these payoffs cannot be homeomorphic to a circle - thus, ruling out the Levy-McLennan type of example.

If, for example, a DSG satisfying assumptions [DSG-1] has Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, such that for each ω, $\mathcal{N}(\omega,\cdot)$ contains a convex-valued or star-shape valued sub-USCO, $\eta(\omega,\cdot)$ - and thus, is R_δ-valued - then the DSG is approximable. In addition, if for each ω, $\mathcal{N}(\omega,\cdot)$ contains a sub-USCO, $\eta(\omega,\cdot)$, taking arc-like continuum values, arc-smooth continuum values, or dendritic values, then $\eta(\omega,\cdot)$ is contractibly-valued - and thus, is R_δ-valued (for related results see Cellina, 1969, De Blasi and Myjak, 1986, and Beer, 1988) - implying that the DSG is approximable.

[18]If Z is a compact metric space, then Z is R_δ provided there is a sequence of *contractible* spaces, $\{Z^n\}_n$, such that

$$Z^{n+1} \subset Z^n$$

for all n with

$$Z = \cap_{n=1}^{\infty} Z^n.$$

Part IV
Appendices

8 Appendix 1: Mathematical Preliminaries

8.1 A Brief Tour of USCOs

A arbitrary correspondence, Γ, is a set-valued mapping,

$$\Gamma : \mathcal{L}_X^\infty \longrightarrow P(X),$$

taking nonempty values. Here $P(X)$ denotes the collection of nonempty subsets of X. The collection of all such mapping is denoted by

$$\mathcal{F}_{\mathcal{L}_X^\infty\text{-}X} := \mathcal{F}(\mathcal{L}_X^\infty, P(X)).$$

We will often restrict attention to closed-valued correspondences,

$$\mathcal{N}(\cdot) : \mathcal{L}_X^\infty \longrightarrow P_{w^*f}(X),$$

defined on $\mathcal{L}_X^\infty$ taking nonempty, w^*-closed (and hence w^*-compact) values in X. Here, $P_{w^*f}(X)$ denotes the collection of nonempty, w^*-closed subsets of X. $\mathcal{N}(\cdot)$ is w^*-w^*-upper semicontinuous if for any $F \in P_{w^*f}(X)$

$$\mathcal{N}^-(F) := \{v \in \mathcal{L}_X^\infty : \mathcal{N}(v) \cap F \neq \varnothing\} \in P_{w^*f}(\mathcal{L}_X^\infty),$$

where $P_{w^*f}(\mathcal{L}_X^\infty)$ denotes the collection of nonempty, w^*-closed subsets of $\mathcal{L}_X^\infty$.[19] Following the literature (e.g., see Crannell, Franz, and LeMasurier, 2005, Hola and Holy, 2009) such a set-valued mapping is called an USCO. The collection of all such mapping is denoted by

$$\mathcal{U}_{w^*\text{-}w^*} := \mathcal{U}(\mathcal{L}_X^\infty, P_{w^*f}(X)).$$

Denote by $Gr\mathcal{N}$ the graph of $\mathcal{N}(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$ given by

$$Gr\mathcal{N} := \{(v, x) \in \mathcal{L}_X^\infty \times X : x \in \mathcal{N}(v)\}.$$

An USCO $\eta(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$ is minimal if $\varphi(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$ and $Gr\varphi \subseteq Gr\eta$ implies that $Gr\varphi = Gr\eta$. Denote by

$$\mathcal{M}_{w^*\text{-}w^*} := \mathcal{M}(\mathcal{L}_X^\infty, P_{w^*f}(X))$$

the collection of all minimal USCOs. Each USCO contains at least one minimal USCO (e.g., see Proposition 4.3 in Drewnowski and Labuda, 1990).

An USCO $\phi(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$ is a sub-USCO belonging to $\mathcal{N}(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$ if $Gr\phi \subseteq Gr\mathcal{N}$. The collection of all sub-USCOs belonging to $\mathcal{N}(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$ is denoted by

$$\mathcal{U}_{w^*\text{-}w^*}[\mathcal{N}(\cdot)].$$

Let $[\mathcal{N}(\cdot)]$ denote the collection of all minimal USCOs belonging to $\mathcal{N}(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$. Thus,

$$[\mathcal{N}(\cdot)] := \{\eta(\cdot) \in \mathcal{M}_{w^*\text{-}w^*} : Gr\eta \subseteq Gr\mathcal{N}\}.$$

[19]Equivalently, $\mathcal{N}(\cdot)$ is W^*-w^*-upper semicontinuous (usc) if given any w^*-open subset G of X, the set

$$\mathcal{N}^+(G) := \{v \in \mathcal{L}_X^\infty : \mathcal{N}(v) \subset G\}$$

is W^*-open in $\mathcal{L}_X^\infty$.

An USCO, $\mathcal{N}(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$, such that

$$[\mathcal{N}(\cdot)] = \{\eta(\cdot)\} \text{ for some } \eta(\cdot) \in \mathcal{M}_{w^*\text{-}w^*}\},$$

is called a quasi-minimal USCO. Let $\mathcal{QM}_{w^*\text{-}w^*}$ denote the collection of all quasi-minimal USCOs. Note that for any $\mathcal{N}(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$, each minimal USCO belong to $\mathcal{N}(\cdot)$ is quasi-minimal. Thus,

$$[\mathcal{N}(\cdot)] \subseteq \mathcal{QM}_{w^*\text{-}w^*}.$$

Finally, given any USCO $\mathcal{N}(\cdot) \in \mathcal{U}_{w^*\text{-}w^*}$ let

$$S(\mathcal{N}) := \{v \in \mathcal{L}_X^\infty : \mathcal{N}(v) = \{x\} \text{ for some } x \in X\}$$

denote the set of points in $\mathcal{L}_X^\infty$ where $\mathcal{N}(\cdot)$ is single valued. Because $\mathcal{L}_X^\infty$ is a compact metric Baire space and X is metrizable (in this case with metric ρ_{w^*}), if $\mathcal{N}(\cdot) \in \mathcal{QM}_{w^*\text{-}w^*}$, then $S(\mathcal{N})$ is a dense G_δ set in $\mathcal{L}_X^\infty$ (see Lemma 7 in Anguelov and Kalenda, 2009).

We will denote by

$$\mathcal{U}_{w^*\text{-}W^*} := \mathcal{U}(\mathcal{L}_X^\infty, P_{w^*f}(\mathcal{L}_X^\infty))$$

the collection of all w^*-w^*-upper semicontinuous correspondences defined on $\mathcal{L}_X^\infty$ taking values in $P_{w^*f}(\mathcal{L}_X^\infty)$, the hyperspace of nonempty, w^*-closed subsets of the w^*-compact, convex subset, $\mathcal{L}_X^\infty$, of the separable norm dual, $\mathcal{L}_{E^*}^\infty$, of the separable Banach space $\mathcal{L}_E^1$.

8.2 Upper Caratheodory Correspondences, Decomposability, and Selections

A correspondence, $\mathcal{P}(\cdot,\cdot)$, taking values in $P_{w^*f}(X)$ is upper Caratheodory on $\Omega \times \mathcal{L}_X^\infty$ if it is jointly measurable in ω and v and upper semicontinuous in v. We will denote by

$$\mathcal{UC}_{w^*\text{-}w^*} := \mathcal{UC}(\Omega \times \mathcal{L}_X^\infty, P_{w^*f}(X)),$$

the collection of all upper Caratheodory correspondences defined on $\Omega \times \mathcal{L}_X^\infty$ taking nonempty w^*-closed values in X.

Each upper Caratheodory correspondence, $\mathcal{P}(\cdot,\cdot)$, has associated with it four other correspondences which will be of interest to us in solving our fixed point problem. First, there is for each $v \in \mathcal{L}_X^\infty$ the measurable correspondence,

$$\omega \longrightarrow \mathcal{P}(\omega, v) := \mathcal{P}_v(\omega),$$

which takes nonempty, weak*-closed values in $X \subset E^*$. We will sometimes refer to the induced collection of measurable correspondences,

$$\mathcal{P}^{(B_\Omega, B_{w^*})} := \{\mathcal{P}(\cdot, v) \in \mathcal{M}_{B_\Omega\text{-}B_{w^*}} : v \in \mathcal{L}_X^\infty\}.$$

as the measurable part of $\mathcal{P}(\cdot,\cdot)$. Second, there is for each $\omega \in \Omega$, the w^*-w^*-upper semicontinuous correspondence,

$$v \longrightarrow \mathcal{P}(\omega, v) := \mathcal{P}_\omega(v),$$

which also takes nonempty, weak*-closed values in $X \subset E^*$. We will sometimes refer to the induced collection of upper semicontinuous correspondences,

$$\mathcal{P}^{USCO} := \{\mathcal{P}(\omega, \cdot) \in \mathcal{U}_{w^*\text{-}w^*} : \omega \in \Omega\},$$

as the USCO part (or the upper semicontinuous part) of $\mathcal{P}(\cdot,\cdot)$.[20] Third, there is the graph correspondence,

$$\omega \longrightarrow Gr\mathcal{P}(\omega,\cdot) \in P_{w^*\text{-}w^*f}(\mathcal{L}_X^\infty \times X),$$

given by

$$\omega \longrightarrow Gr\mathcal{P}(\omega,\cdot) := \{(v,U) \in \mathcal{L}_X^\infty \times X : U \in \mathcal{P}(\omega,v)\},$$

a mapping from the state space into the graphs of the USCO part of $\mathcal{P}(\cdot,\cdot)$. Under assumptions [A-1], it follows from Lemma 3.1(ii) in Kucia and Nowak (2000) that the graph correspondence,

$$\omega \longrightarrow Gr\mathcal{P}(\omega,\cdot) := Gr\mathcal{P}_\omega(\cdot),$$

is $(B_\Omega, B_{w^*} \times B_{w^*})$-measurable.

Finally, there is the induced measurable-selection-valued correspondence,

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot,v)) := \mathcal{S}^\infty(\mathcal{P}_v),$$

with values given by,

$$\mathcal{S}^\infty(\mathcal{P}_v) := \left\{U_{(\cdot)} \in \mathcal{L}_X^\infty : U_\omega \in \mathcal{P}(\omega,v) \text{ a.e. } [\mu]\right\}.$$

For each $v \in \mathcal{L}_X^\infty$, $\mathcal{S}^\infty(\mathcal{P}_v)$ is the set of all μ-equivalence classes of functions $U_{(\cdot)} \in \mathcal{L}_X^\infty$ such that $U_\omega \in \mathcal{P}(\omega,v)$ a.e. $[\mu]$. We will refer to the set $\mathcal{S}^\infty(\mathcal{P}_v)$ for each $v \in \mathcal{L}_X^\infty$ as the set of μ-equivalence classes of $\mathcal{L}_X^\infty$-measurable, selections of $\mathcal{P}_v(\cdot)$.

A subset $\mathcal{S}$ of $\mathcal{L}_X^\infty$ is said to be *decomposable* if for any two functions $U^0_{(\cdot)}$ and $U^1_{(\cdot)}$ in $\mathcal{S}$ and for any $E \in B_\Omega$, we have

$$U^0_{(\cdot)} I_E(\cdot) + U^1_{(\cdot)} I_{\Omega\backslash E}(\cdot) \in \mathcal{S}.$$

For any upper Caratheodory correspondence, $\mathcal{P}(\cdot,\cdot) : \Omega \times \mathcal{L}_X^\infty \longrightarrow P_{w^*f}(X)$, we have for each $v \in \mathcal{L}_X^\infty$, that the induced measurable-selection-valued correspondence, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, takes decomposable values.

We will denote by $\Sigma^\infty(\mathcal{P}_v)$ the prequotient of $\mathcal{S}^\infty(\mathcal{P}_v)$ (i.e., the set of *all* a.e. measurable selections of $\mathcal{P}_v(\cdot)$) and we will denote by $\Sigma(\mathcal{P}_v)$ the set of *all* (everywhere) measurable selections of $\mathcal{P}_v(\cdot)$.[21]

9 Appendix 2: w^*-Convergence and K-Convergence

9.1 w^*-Convergence in $\mathcal{L}_{R^m}^\infty$

Equip $\mathcal{L}_{R^m}^\infty$ with the weak star topology (i.e., the w^*-topology or the $\sigma(\mathcal{L}_{R^m}^\infty, \mathcal{L}_{R^m}^1)$-topology) and denote by $L_{R^m}^\infty$ the prequotient space of all measurable functions defined

[20] Following the literature (e.g., Hola and Holy, 2009), because the mapping, $\nu \longrightarrow \mathcal{P}_\omega(v)$, takes nonempty, w^*-compact values we will refer to such mappings as USCOs and we will denote the collection of all W^*-w^*-USCOs by

$$\mathcal{U}_{W^*w^*} := \mathcal{U}(\mathcal{L}_X^\infty, P_{w^*f}(X)).$$

[21] Because $\mathcal{L}_X^\infty$ and $\mathcal{S}(\mathcal{P}_v)$ are spaces of μ-equivalence classes, the notation $U_{(\cdot)} \in \mathcal{L}_X^\infty$ or $U_{(\cdot)} \in \mathcal{S}(\mathcal{P}_v)$ means $[U_{(\cdot)}] \in \mathcal{L}_X^\infty$ or $[U_{(\cdot)}] \in \mathcal{S}(\mathcal{P}_v)$, where $U_{(\cdot)} \in L_X^\infty$ or $U_{(\cdot)} \in \Sigma(\mathcal{P}_v)$, and

$$[U_{(\cdot)}] := \{U'_{(\cdot)} \in L_X^\infty : U'_\omega = U_\omega \text{ a.e. } [\mu]\}.$$

Thus, $U_{(\cdot)} \in \mathcal{S}(\mathcal{P}_v)$ means that $U_{(\cdot)} \in \mathcal{L}_X^\infty$ and for all $U'_{(\cdot)} \in [U_{(\cdot)}]$

$$U'_\omega \in \mathcal{P}_v(\omega) \text{ a.e. } [\mu].$$

on Ω with values in R^m. We have the following definitions:

Definitions A2.1
(1) (Weak Star Convergence in $\mathcal{L}^\infty_{R^m}$) A sequence of R^m-valued functions $\{v^n\}_n$ in $\mathcal{L}^\infty_{R^m}$ converges weak star, or w^*-converges to a function $v^* \in \mathcal{L}^\infty_{R^m}$ if for every function $l \in \mathcal{L}^1_{R^m}$,

$$\langle v^n, l \rangle := \int_\Omega \langle v^n(\omega), l(\omega) \rangle \, d\mu(\omega) \longrightarrow \int_\Omega \langle v^*(\omega), l(\omega) \rangle \, d\mu(\omega) := \langle v^*, l \rangle \, . \tag{23}$$

(2) (Sequential Weak Star Compactness in $\mathcal{L}^\infty_{R^m}$) A subset $\mathcal{H}$ of (μ-equivalence classes) of functions in $\mathcal{L}^\infty_{R^m}$ is sequentially weak star compact if every sequence $\{v^n\}_n$ in $\mathcal{H}$ has a weak star converging subsequence $\{v^{n_k}\}_k$ with limit v^* contained in $\mathcal{H}$.

Note that w^*-convergence is with respect to μ-equivalence classes.

9.2 K-Convergence in $L^1_{R^m}$

Consider a sequence $\{U^n(\cdot)\}_n \subset L^1_{R^m}$ with corresponding sequence of arithmetic mean functions, $\left\{\frac{1}{n}\sum_{k=1}^n U^k(\cdot)\right\}_n$, and for any subsequence, $\{U^{n_k}(\cdot)\}_k$, of $\{U^n(\cdot)\}_n$, let the corresponding subsequence of arithmetic mean functions be given by

$$\left\{\frac{1}{k}\sum_{i=1}^{k} U^{n_i}(\cdot)\right\}_n .$$

Finally, for each n, let $\widehat{U}^n(\cdot) := \frac{1}{n}\sum_{k=1}^n U^k(\cdot)$ and for each k, let $\widehat{U}^{n_k}(\cdot) := \frac{1}{k}\sum_{i=1}^k U^{n_i}(\cdot)$.

Definition A2.2 (*K-Sequences, K-Convergence, and K-Limits*)
We say that a sequence $\{U^n\}_n \subset L^1_{R^m}$ is K-convergent (or is a K-sequence) if there exists a K-limit function $\widehat{U} \in L^1_{R^m}$ such that,
(a) the corresponding sequence of arithmetic mean functions, $\{\widehat{U}^n(\cdot)\}_n$, converges pointwise a.e. $[\mu]$ to $\widehat{U}(\cdot)$, that is,

$$\widehat{U}^n(\omega) \longrightarrow \widehat{U}(\omega) \text{ a.e. } [\mu],$$

and
(b) for any subsequence, $\{U^{n_k}(\cdot)\}_k$, of $\{U^n(\cdot)\}_n$, the corresponding subsequence of arithmetic mean functions, $\{\widehat{U}^{n_k}(\cdot)\}_n$, converges pointwise a.e. $[\mu]$ to $\widehat{U}(\cdot)$ as well, that is,

$$\widehat{U}^{n_k}(\omega) \longrightarrow \widehat{U}(\omega) \text{ a.e. } [\mu].$$

We will often refer to set of μ-measure zero where pointwise arithmetic mean convergence fails for a particular subsequence as the subsequence's *exceptional set.*

We say that a set of functions, $\mathcal{H} \subset \mathcal{L}^1_{R^m}$, is K-compact if every sequence, $\{U^n\}_n \subset \mathcal{H}$, has a K-convergent subsequence with K-limit contained in $\mathcal{H}$. By Komlos' Theorem any $\|\cdot\|_1$-bounded subset $\mathcal{H}$ of $\mathcal{L}^1_{R^m}$ is relatively K-compact (i.e., has a K-converging subsequence with K-limit contained in $\mathcal{L}^1_{R^m}$).

A sequence, $\{U^n(\cdot)\}_n$, of functions in $\mathcal{L}^1_{R^m}$ is norm-bounded provided

$$\sup_n \|U^n\|_1 := \sup_n \sum_{d=1}^m \|U^n_d\|_1 < \infty.$$

For the convenience of the reader we state the Theorem of Komlos (1967) as well as Page's (1991) lower closure result.

Komlos Theorem (1967):
If $\{U^n(\cdot)\}_n \subset \mathcal{L}^1_{R^m}$ *is* $\|\cdot\|_1$*-bounded, then* $\{U^n(\cdot)\}_n$ *has a* K*-convergent subsequence.*
Page's Theorem (1991):
If the sequence $\{v^n(\cdot)\}_n \subset \mathcal{L}^1_{R^m}$ *is* $\|\cdot\|_1$*-bounded and K-converges to some integrable* R^m*-valued function,* $\widehat{v}(\cdot)$*, then*

$$\widehat{v}(\omega) \in coLs\{v^n(\omega)\} \text{ a.e. } [\mu]$$

and there exists an integrable R^m*-valued function,* $v^*(\cdot)$*, such that* $v^*(\omega) \in Ls\{v^n(\omega)\}$ *a.e.* $[\mu]$ *and*

$$\int_\Omega v^*(\omega)d\mu(\omega) = \int_\Omega \widehat{v}(\omega)d\mu(\omega).$$

9.3 K-convergence and w^*-Convergence in $\mathcal{L}^\infty_{R^m}$

Our next results concern the relationships between K-convergence and weak star (w^*-convergence) in $\mathcal{L}^\infty_{R^m}$.

Theorem A2.1 *(K-Convergence and w^*-Convergence):*
Suppose the primitives satisfy assumptions [DSG-1]. Let $\{v^n\}_n$ *be any sequence in* $\mathcal{L}^\infty_{R^m}$*. Then the following statements are true:*
(1) If $\{v^n\}_n$ *K-converges to* $\widehat{v} \in L^\infty_{R^m}$*, then* $\{v^n\}_n \subset \mathcal{L}^\infty_{R^m}$ w^**-converges to* $\widehat{v} \in \mathcal{L}^\infty_{R^m}$*.*
(2) If $\{v^n\}_n \subset \mathcal{L}^\infty_{R^m}$ w^**-converges to* $v^* \in \mathcal{L}^\infty_{R^m}$*, then each K-convergent subsequence of* $\{v^n\}_n$ *has a K-limit,* $\widehat{v} \in L^\infty_{R^m}$*, such that* $\widehat{v} = v^*$ *a.e.* $[\mu]$*.*

Before proceeding to the proof, some comments on notation: In the statement of the Theorem above, we write $\{v^n\}_n \subset \mathcal{L}^\infty_{R^m}$, to indicate that rather than viewing the sequence $\{v^n\}_n$ as a sequence of specific functions - which we will denote by $\{v^n\}_n \subset L^\infty_{R^m}$ - we are instead viewing the sequence as a sequence of μ-equivalence classes in $\mathcal{L}^\infty_{R^m}$ indexed by the specific functions, v^n. Thus, we write $\widehat{v} \in \mathcal{L}^\infty_{R^m}$ to denote the μ-equivalence class in $\mathcal{L}^\infty_{R^m}$ determined by the specific function, $\widehat{v}$.

PROOF: We will prove part (2) first. Assume that $\{v^n\}_n \subset \mathcal{L}^\infty_{R^m}$ w^*-converges to $v^\infty \in \mathcal{L}^\infty_{R^m}$, and that the subsequence, $\{v^{n_k}\}_k$, K-converges to $\widehat{v} \in L^\infty_{R^m}$. For each N and each $l \in L^1_{R^m}$ we have

$$\frac{1}{k}\sum_{i=1}^{k} v^{n_i}(\omega)l(\omega) \longrightarrow \widehat{v}(\omega)l(\omega) \text{ a.e. } [\mu]$$

and by the Dominated Convergence Theorem also in $\mathcal{L}^1_{R^m}$-norm. Thus, for each $l \in L^1_{R^m}$,

$$\int_\Omega \widehat{v}(\omega)l(\omega)d\mu(\omega)$$

$$:= \lim_{k\longrightarrow\infty} \tfrac{1}{k}\sum_{i=1}^{k} \int_\Omega v^{n_i}(\omega)l(\omega)d\mu(\omega)$$

$$= \int_\Omega v^\infty(\omega)l(\omega)d\mu(\omega),$$

and hence $\widehat{v}(\omega) = v^\infty(\omega)$ a.e. $[\mu]$.

Now we will prove part (1). Assume that $\{v^n\}_n \subset L^\infty_{R^m}$ K-converges to some $\widehat{v} \in L^\infty_{R^m}$. In order to show that $\{v^n\}_n$ w^*-converges to $\widehat{v}$, by w^*-compactness and metrizability, it suffices to show that the μ-equivalence class in $\mathcal{L}^\infty_{R^m}$ containing $\widehat{v}$ is the only limit point of the sequence of μ-equivalence classes, $\{v^n\}_n \subset \mathcal{L}^\infty_{R^m}$. Let v^∞ be any w^*-limit point of the sequence $\{v^n\}_n$ and let $\{v^{n_k}\}_k$ be a subsequence w^*-converging to v^∞. By K-convergence we know that this subsequence also K-converges to $\widehat{v}$ and hence by part (2) we know that $v^\infty = \widehat{v}$ a.e. $[\mu]$. Q.E.D.

9.4 w^*-Compactness and Metrizability of $\mathcal{L}_X^\infty$ for the w^* Topology

We are now in a position to prove our preliminary result on the w^*-compactness and metrizability of $\mathcal{L}_X^\infty$. By Theorem V.1 in Castaing and Valadier (1977) (also see, Theorem 7.14 in Kahn, 1985), the space $\mathcal{L}_X^\infty$ of all equivalence classes of (B_Ω, B_{R^m})-measurable functions taking values in X a.e. $[\mu]$ is a w^*-compact subset of $\mathcal{L}_{R^m}^\infty$.

Theorem A2.2 *(Metrizability of $\mathcal{L}_X^\infty$)*
The convex set $\mathcal{L}_X^\infty$ of all equivalence classes of (B_Ω, B_{R^m})-measurable functions taking values in X a.e. $[\mu]$ is compact and metrizable for the weak star topology.

PROOF: By Theorem 6.30 in Aliprantis and Border (2006), it suffices to show that (the quotient) $\mathcal{L}_E^1$ is (norm) separable. Recall (i) that the σ-field, B_Ω, is countably generated and (ii) that R^m is separable. Hence, let $\{A_j\}_{j=1}^\infty$ be the countable collection of subsets of Ω generating $B(\Omega)$, and let $\{x_i\}_{i=1}^\infty$ be a countable dense subset of R^m. Note that the set of vectors $\{x_i\}_{i=1}^\infty$ in R^m separates the vectors in R^m (i.e., for p and p' in R^m, $\langle p, x_i\rangle = \langle p', x_i\rangle$ for all i implies that $p = p'$), and for each i and j define $l_{ij}(\cdot) \in L_{R^m}^1$ as follows: $l_{ij}(\omega) := x_i I_{A_j}(\omega)$ where $I_{A_j}(\cdot)$ is the indicator function for the set A_j. For $v \in L_{R^m}^\infty$, let $\langle v, l_{ij}\rangle$ be given by

$$\langle v, l_{ij}\rangle := \textstyle\int_\Omega \langle v(\omega), l_{ij}(\omega)\rangle \, d\mu(\omega) = \int_\Omega \left\langle v(\omega), x_i I_{A_j}(\omega)\right\rangle d\mu(\omega).$$

Finally, observe that the set of functions $\{l_{ij} : i \text{ and } j \text{ in } \mathbb{N}\} \subset L_{R^m}^1$ is (norm) dense in $\mathcal{L}_{R^m}^1$ and separates the functions in $L_{R^m}^\infty$, i.e., if

$$\langle v, l_{ij}\rangle = \langle v', l_{ij}\rangle \text{ for all } i \text{ and } j, \text{ then } v(\omega) = v'(\omega) \text{ a.e. } [\mu].$$

Because $\mathcal{L}_X^\infty$ is $\|\cdot\|^*$-bounded, given the separability of $\mathcal{L}_{R^m}^1$ by Theorem 6.30 in Aliprantis and Border (2006) $\mathcal{L}_X^\infty \subset \mathcal{L}_{R^m}^\infty$ is metrizable for the w^*-topology in $\mathcal{L}_{R^m}^\infty$ (i.e., the $\sigma(\mathcal{L}_{R^m}^\infty, \mathcal{L}_{R^m}^1)$ topology). Thus, $\mathcal{L}_X^\infty$ is w^*-compact and metrizable for the relative w^*-topology inherited from $\mathcal{L}_{R^m}^\infty$. QED

To fix the notation, let ρ_{w^*} be a metric compatible with the relative w^*-topology (the $\sigma(\mathcal{L}_{R^m}^\infty, \mathcal{L}_{R^m}^1)$ topology) on $\mathcal{L}_X^\infty$ and let ρ_X a metric on $X \subset R^m$.

10 Appendix 3: Continuity of Payoff Functions

Theorem A3.1 *(Continuity of Payoff Functions):*
Let DSG be a discounted stochastic game satisfying assumptions [DSG-1] with players' payoff profile function,

$$(v, a) \longrightarrow U(\omega, a, v) := (U_1(\omega, a, v_1), \ldots, U_m(\omega, a, v_m)).$$

If $\{(v^n, a^n)\}_n$ is a sequence in $\mathcal{L}_X^\infty \times A$ such that $v^n \underset{w^*}{\longrightarrow} v^*$ and $a^n \underset{\rho_A}{\longrightarrow} a^*$, then in each state $\omega \in \Omega$,

$$U(\omega, a^n, v^n) \underset{R^m}{\longrightarrow} U(\omega, a^*, v^*).$$

PROOF: Let $\{(v^n, a^n)\}_n$ be a sequence such that $v^n \underset{w^*}{\longrightarrow} v^*$ and $a^n \underset{\rho_A}{\longrightarrow} a^*$. Let ω be given and fixed, and observe that for each players d:

$$|U_d(\omega, a^n, v_d^n) - U_d(\omega, a^*, v_d^*)|_R$$

$$\leq \underbrace{|U_d(\omega, a^n, v_d^n) - U_d(\omega, a^*, v_d^n)|_R}_{A^n} + \underbrace{|U_d(\omega, a^*, v_d^n) - U_d(\omega, a^*, v_d^*)|_R}_{B^n}.$$

We will carry out our proof for one player d, keeping in mind that the argument holds for all players simultaneously. Consider B^n first. We have

$$B^n = \beta_d \left| \int_\Omega v_d^n(\omega')q(\omega'|\omega,a^*) - \int_\Omega v_d^*(\omega')q(\omega'|\omega,a^*)\right|_R .$$

Let $h(\cdot|\omega,a^*)$ be a density of $q(\cdot|\omega,a^*)$ with respect to μ. Given that $v_d^n \underset{w_d^*}{\longrightarrow} v_d^*$, we have,

$$\begin{aligned}&\textstyle\int_\Omega v_d^n(\omega')q(\omega'|\omega,a^*) = \int_\Omega v_d^n(\omega')h(\omega'|\omega,a^*)d\mu(\omega')\\ &\longrightarrow \textstyle\int_\Omega v_d^*(\omega')h(\omega'|\omega,a^*)d\mu(\omega') = \int_\Omega v_d^*(\omega')q(\omega'|\omega,a^*).\end{aligned}$$

Thus, $B^n \xrightarrow{n} 0$.

Next, consider A^n. We have

$$A^n \leq (1-\beta_d)\underbrace{|r_d(\omega,a^n) - r_d(\omega,a^*)|_R}_{A_1^n}$$

$$+\underbrace{\beta_d \left| \int_\Omega v_d^n(\omega')q(\omega'|\omega,a^n) - \int_\Omega v_d^n(\omega')q(\omega'|\omega,a^*)\right|_R}_{A_2^n}.$$

Continuity of $r_d(\omega,\cdot)$ and $a^n \underset{\rho_A}{\longrightarrow} a^*$ imply that $A_1^n \xrightarrow{n} 0$. To see that $A_2^n \xrightarrow{n} 0$, observe that

$$\begin{aligned}&\left|\textstyle\int_\Omega v_d^n(\omega')q(\omega'|\omega,a^n) - \int_\Omega v_d^n(\omega')q(\omega'|\omega,a^*)\right|\\ &\leq M \left\|q(\cdot|\omega,a^n) - q(\cdot|\omega,a^*)\right\|_\infty \xrightarrow{n} 0. \quad \text{Q.E.D.}\end{aligned}$$

11 Appendix 4: Metric Topology

11.1 Basics

Throughout assume that (Z,ρ_z) and (X,ρ_X) are compact metric spaces.[22] Because the space Z is compact, for any collection $\{G_\alpha\}_\alpha$ of open sets in Z where $Z = \cup_\alpha G_\alpha$ and α ranges over an arbitrary set A, there exists a finite subcollection, $\alpha_1,\ldots,\alpha_n$ such that $Z = G_{\alpha_1}\cup\cdots\cup G_{\alpha_n}$ (i.e., the Borel-Lebesgue condition - every open cover of Z contains a finite subcover). The Borel-Lebesgue condition is equivalent to the Riesz condition: if $\{F_\alpha\}_\alpha$ is a collection of closed sets in Z such that $\cap_\alpha F_\alpha = \emptyset$, then there is a finite subcollection, $\alpha_1,\ldots,\alpha_n$ such that $F_{\alpha_1}\cap\cdots\cap F_{\alpha_n} = \emptyset$ (see Kuratowski, 1972).

Let $\mathcal{C}(Z,X)$ denote the collection of continuous functions defined on Z taking values in X. If $f \in \mathcal{C}(Z,X)$ is *one-to-one*, from Z *onto* X, and if its *inverse,* f^{-1}, *is also continuous*, then we say that f is a homeomorphism and that the metric spaces Z and X are homeomorphic. If (Z,ρ_Z) is compact, then any continuous, one-to-one mapping f from Z onto X is a homeomorphism. A function, $f : Z \longrightarrow X$ is an embedding if $f : Z \longrightarrow f(Z)$ is a homeomorphism. In this case we can think of Z as a topological subspace of X by identifying Z with its image $f(Z)$.

[22] More detail on the topics covered in this Appendix can be found in Willard (1970) and Illanes and Nadler (1999).

11.2 Continua

Given metric space (Z, ρ_Z), a set $E \subseteq Z$ is *connected* if E cannot be written as the union of two disjoint open sets (or two disjoint closed sets). A set $E \subseteq Z$ is *locally connected* at $e \in E$ if each neighborhood U_e of e *contains* a connected neighborhood V_e of e. E is locally connected if it is locally connected at each $e \in E$.[23]

If the metric space, (Z, ρ_Z), is compact and connected it is called a *continuum.* Given any continuum, (Z, ρ_Z), a point $z \in Z$ is called a cut point of Z if $Z \backslash \{z\}$ is not connected. A nonempty closed, connected subset of Z is called a *subcontinuum.* If in addition, the continuum, (Z, ρ_Z), is locally connected it is called a *Peano continuum.*

A subset, C, of metric space (Z, ρ_Z) is called an *n-cell* if it is homeomorphic to $I^n := \prod_{i=1}^{n}[0,1]_i := [0,1]^n$. If in particular, C is homeomorphic to the interval $[0,1]$ it is called an *arc* (i.e., an arc, then, is a 1-cell). An *end point* of arc C is either one of the two points of C that are the images of the end points of $[0,1]$ under *any* homeomorphism of $[0,1]$ onto C. A continuum Z is *arcwise connected* if any two points, z^1 and z^2, in Z can be joined by an arc in Z with endpoints z^1 and z^2.

We close this subsection by noting that in any metric space (Z, ρ_Z) the condition of being (i) a locally connected continuum (i.e., a Peano continuum) and (ii) the continuous image of an interval are equivalent (this is the Mazurkiewicz-Moore Theorem - see Kuratowski, 1972). Thus, a Peano continuum (with or without an M-convex metric) is the continuous image of the unit interval, $[0,1]$.

11.3 Homotopies

We begin by recalling the notion of a homotopy - a function that essentially provides us with a way to index a set of continuous functions.

Definition A4.1 *(Homotopies) Let $C(Z \times [0,1], X)$ denote the collection of all continuous functions, $h : Z \times [0,1] \longrightarrow X$, defined on $Z \times [0,1]$ taking values in X. A function $h \in C(Z \times [0,1], X)$ is called a homotopy and each homotopy specifies an index set of continuous functions,*

$$\mathbb{H}_h(Z, X) := \{h(\cdot, t) : t \in [0,1]\}.$$

The indexed collection, $\mathbb{H}_h(Z, X)$, can be thought of as an arc, α_h, in the continuum of continuous functions, $\mathcal{C}(Z, X)$, equipped with the sup metric. The continuous functions, f and g in $\mathcal{C}(Z, X)$ are homotopically related or homotopic, if f and g are the endpoints of an arc α_h whose arc type is identified by some function, $h \in \mathcal{C}(Z \times [0,1], X)$, called a homotopy. In particular, if $f, g \in \mathcal{C}(Z, X)$ are homotopic, then there is an arc of type $h \in \mathcal{C}(Z \times [0,1], X)$ running from continuous function $f(\cdot) = h(\cdot, 0)$ to continuous function $g(\cdot) = h(\cdot, 1)$. We denote this h-arc from f to g by writing $g \in [f]_h$ or by writing $f \xrightarrow{h} g$ (and if the orientation is in the opposite direction, then we write $f \in [g]_h$ or $g \xrightarrow{h} g$). Constant functions form a special class of homotopy arc end points. Let $g_{\overline{x}} \in \mathcal{C}(Z, X)$

[23] Local connectedness differs from connectedness. To see this, note for example that the set E in R given by

$$E = [0,1) \cup (1,2]$$

is locally connected but not connected (because E is equal to the union of two disjoint, half open intervals). While the set G in R^2 given by

$$G := \{(x,0), (x, \frac{1}{n}) : 0 \leq x \leq 1 \text{ and } n = \pm 1, \pm 2, \ldots\} \cup \{(0,y), (1,y) : y \in R\}$$

is connected but not locally connected (because only the point $(0,0)$ and $(1,0)$ in G possess a collection of connected neighborhoods). These examples are taken from Willard (1970), Chapter 8.

denote the constant function (i.e., $g_{\overline{x}}(z) = \overline{x}$ for all $z \in Z$). If f and $g_{\overline{x}}$ are homotopic (i.e., if $g_{\overline{x}} \in [f]_h$, that is, if $f \xrightarrow{h} g_{\overline{x}}$ for some $\overline{x} \in X$), then f is said to be *inessential.* Moreover, if for some pair of compact metric spaces, (Z, ρ_z) and (X, ρ_X), all pairs of functions, $f, g \in \mathcal{C}(Z, X)$, are homotopic, then in particular, $f, g_{\overline{x}} \in \mathcal{C}(Z, X)$, are homotopic for some h-arc and some $\overline{x} \in X$ - and this means that for this pair of compact metric spaces, (Z, ρ_z) and (X, ρ_X), all functions , $f \in \mathcal{C}(Z, X)$, are inessential (i.e., for each $f \in \mathcal{C}(Z, X)$, there is $(h(\cdot, \cdot), x) \in (\mathcal{C}(Z \times [0, 1], X), X)$, $f \xrightarrow{h} g_{\overline{x}}$).

11.4 AR-Spaces and ANR-Spaces

A space Z is an absolute retract, denoted $Z \in AR$, if whenever Z is embedded in some a metric space, say X, then the embedded copy, $f(Z)$, of Z in X - with homeomorphism $f : Z \longrightarrow f(Z) \subset X$, is a retract of X. A space Z is an absolute neighborhood retract, denoted $Z \in ANR$, if whenever Z is embedded in some a metric space, say X, then the embedded copy, $f(Z)$, of Z in X - with homeomorphism $f : Z \longrightarrow f(Z) \subset X$, is a retract of some neighborhood of $f(Z)$ in X.

11.5 Contractible Spaces

If $Z \subseteq X$, then Z is *contractible* in X if for some homotopy $h \in \mathcal{C}(Z \times [0, 1], X)$, there is an h-arc running from the identity (or inclusion) mapping, $f_{id} \in \mathcal{C}(Z, X)$ to a constant mapping, $g_{\overline{x}} \in \mathcal{C}(Z, X)$, for some $\overline{x} \in X$. Thus, $f_{id}(\cdot) = h(\cdot, 0)$ where $f_{id}(z) = z$ for all $z \in Z$ is the inclusion mapping (i.e., $f_{id}(z) = z = h(z, 0)$ for all $z \in Z$) and $h(\cdot, 1)$ is the constant mapping (i.e., $h(z, 1) = \overline{x}$ for all $z \in Z$ for some $\overline{x} \in X$).

We say that X is contractible if X is contractible in X. Note that if X is contractible, then for any $Z \subseteq X$, Z is contractible in X. By far the most useful facts related to the contractibility of continua are the following:

(1) If X is contractible and $Z \subseteq X$ is a retraction of X, then Z is also contractible. Thus if $r : X \xrightarrow{\text{onto}} Z$, $r \in \mathcal{C}(X, Z)$ where $r(z) = z$ for all $z \in Z$, then Z is also contractible.

(2) If X is contractible, then X is unicoherent (see Corollary A.12.10 in van Mill, 2001) - implying that all pairs of functions, $f, g \in \mathcal{C}(X, S^1)$, are homotopic, for the unit circle, $S^1 := \left\{ x = (x_1, x_2) : (x_1)^2 + (x_2)^2 = 1 \right\}$. Thus, if X is contractible, then all continuous functions, $f : X \longrightarrow S^1$ are inessential and we can conclude that X contains no simple closed curves.

11.6 R_δ-Spaces

A space Z is called an R_δ-space, denoted $Z \in R_\delta$, if there exists a sequence of compact, nonempty AR spaces, $\{X^n\}_n$ such that

$$X^{n+1} \subseteq X^n \text{ for every } n$$
$$\text{and}$$
$$X = \cap_{n=1}^{\infty} X^n.$$

If Z is compact, then we have the following inclusion ordering over the topological properties of Z:

$$AR \ \subset \text{contractible} \subset R_\delta.$$

Note that if Z is an AR space, it is an ANR space.

References

[1] Aliprantis, C. D. and Border, K. C. (2006) *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3rd Edition, Springer-Verlag, Berlin-Heidelberg.

[2] Amir, R. (1989) "A Lattice-Theoretic Approach to a Class of Dynamic Games," *Computers and Mathematical Applications* 17, 1345-1349.

[3] Amir, R. (1991) "On Stochastic Games with Uncountable State and Action Spaces," In T. E. S. Raghavan, et al. (eds.), *Stochastic Games and Related Topics in Honor of Professor L. S. Shapley*, Kluwer Academic: Dordrecht.

[4] Amir, R. (1996) "Continuous Stochastic Games of Capital Accumulation with Convex Transitions," *Games and Economic Behavior* 15, 111-131.

[5] Anguelov, R. and Kalenda, O. F. K. (2009) "The Convergence Space of Minimal USCO Mappings," *Czechoslovak Mathematical Journal* 134, 101-128.

[6] Ash R. (1972) *Probability and Real Analysis*, John Wiley & Sons, New York.

[7] Beer, G. (1988) "On a Theorem of Cellina for set Valued Functions," *Rocky Mountain Journal of Mathematics* 18, 37-47.

[8] Billingsley, P. (1986) *Probability and Measure*, 2nd Edition, John Wiley & Sons, New York.

[9] Blackwell, D. (1965) "Discounted Dynamic Programming," *Annals of Mathematical Statistics* 36, 226-235.

[10] Castaing, C. and Valadier (1977) *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics 580, Springer-Verlag, Berlin.

[11] Cellina, A. (1969) "Approximation of Set Valued Functions and Fixed Point Theorems," *Annali di Matematica Pura ed Applicata* Series IV, LXXXI, 17-24.

[12] Chakrabarti, S. (2008) "Pure Strategy Markov Equilibrium in Stochastic Games with Concave Transition Probabilities," Department of Economics, IUPUI, Working Paper.

[13] Crannell, A., Franz, M., and LeMasurier, M. (2005) "Closed Relations and Equivalence Classes of Quasicontinuous Functions," *Real Analysis Exchange* 31, 409-424.

[14] Curtat, L. O. (1996) "Markov Equilibria of Stochastic Games with Complementarities," *Games and Economic Behavior* 17, 177-199.

[15] De Blasi, F. S. and Myjak, J. (1986) "On Continuous Approximations for Multifunctions," *Pacific Journal of Mathematics* 123, 9-31.

[16] Diestel, J. and Uhl, J. J., Jr. (1977) *Vector Measures*, Mathematical Surveys Number 15, American Mathematical Society, Providence, Rhode Island.

[17] Drewnowski, L. and Labuda, I. (1990) "On Minimal Upper Semicontinuous Compact-Valued Maps," *Rocky Mountain Journal of Mathematics* 20, 737-752.

[18] Duggan, J. (2012) "Noisy Stochastic Games," *Econometrica* 80, 2017-2045.

[19] Fryszkowski, A. (1983) "Continuous Selections from a Class of Non-convex Multivalued Maps," *Studia Math.* 76, 163-174.

[20] Fink, A. M. (1964) "Equilibrium in Stochastic n-Person Games," *Journal of Science of the Hiroshima University Series A-I* 28, 89-93.

[21] Federgruen, A. (1978) "On N-Person Stochastic Games with Denumerable State Space," *Advances in Applied Probability* 10, 452-471.

[22] Glicksberg, I. L. (1952) "A Further Generalization of the Kakutani Fixed Point Theorem with Application to Nash Equilibrium Points," *Proceedings of the American Mathematical Society* 3, 170-174.

[23] Gorniewicz, L., Granas, A., and Kryszewski, W. (1991) "On the Homotopy Method in the Fixed Point Index Theory of Multi-Valued Mappings of Compact Absolute Neighborhood Retracts," *Journal of Mathematical Analysis and Applications* 161, 457

[24] Duffie, D., Geanakoplos, J., Mas-Colell, A., and McLennan, A. (1994) "Stationary Markov Equilibria," *Econometrica* 62, 745-781.

[25] Dynkin, E. B., and Evstigneev, I. V. (1976) "Regular Conditional Expectations of Correspondences," *Theory of Probability and Its Applications* 21, 325-338.

[26] Harris, C., Reny, P., and Robson A. (1995) "The Existence of Subgame-Perfect Equilibrium in Continuous Games with Almost Perfect Information: A Case for Public Randomization," *Econometrica* 63, 507-544.

[27] He, W., and Sun Y. (2015) "Stationary Markov Perfect Equilibria in Discounted Stochastic Games," *arXiv:1311.1562.v [math.OC] 6 Nov 2015.*

[28] Hola, L. and Holy, D. (2009) "Minimal USCO Maps, Densely Continuous Forms, and Upper Semicontinuous Functions," *Rocky Mountain Journal of Mathematics* 39, 545-562.

[29] Hildenbrand, W. (1974) *Core and Equilibria of a Large Market*, Princeton University Press, Princeton, New Jersey.

[30] Himmelberg, C. J. (1975) "Measurable Relations," *Fund. Math.* 87, 53-72.

[31] Himmelberg, C. J., and VanVleck, F. S. (1975) "Multifunctions with Values in a Space of Probability Measures," *Journal of Mathematical Analysis and Applications* 50, 108-112.

[32] Himmelberg, C. J., Parthasarathy, T., Raghavan, T. E. S., and VanVleck, F. S. (1976) "Existence of p-equilibrium and Optimal Stationary Strategies in Stochastic Games,"

[33] Horst, U. (2005) "Stationary Equilibria in Discounted Stochastic Games with Weakly Interacting Players," *Games and Economic Behavior* 51, 83-108. *Proceedings of the American Mathematical Society* 60, 245-251.

[34] Illanes, A., and Nadler Jr., S. B. (1999) *Hyperspaces: Fundamentals and Recent Advances*, Marcel Dekker, New York.

[35] Kahn, M. A. (1985) *On Extensions of the Cournot-Nash Theorem*. In: Aliprantis, C. D., Burkinshaw, O., Rothman, N. J. (Eds.), Advances in Equilibrium Theory. Lecture Notes in Economics and Math. Systems 244, Springer-Verlag, 79-106.

[36] Komlos, J. (1967) "A Generalization of a Problem of Steinhaus,"*Acta Mathematica Academiae Scientiarum Hungaricae* 18, 217-229.

[37] Kucia, A. and Nowak, A. (2000) "Approximate Caratheodory Selections," *Bull. Pol. Acad. Sci. Math.* 48, 81-87.

[38] Kuratowski, K. (1972) *Introduction to Set Theory and Topology*, 2nd Edition, PWN - Polish Scientific Publishers - Warszawa.

[39] Kuratowski, K., and Ryll-Nardzewski, C. (1965) "A General Theorem on Selectors," *Bull. Acad. Polon. Sci. S6r. Sci. Math. Astronom. Phys.*, 13, 397-403.

[40] Levy, Y. (2013) "A Discounted Stochastic Game with No Stationary Nash Equilibrium: Two Examples," *Econometrica* 81, 1973-2007.

[41] Levy, Y., and McLennan, A. (2014) "Corrigendum to: "Discounted Stochastic Games with No Stationary Nash Equilibrium: Two Examples" Nuffield College, University of Oxford, UK, OX1 1NF.

[42] Lyapunov, A. (1940) "Sur les Fonctions-Vecteurs Compltement Additives," Bulletin of the Academy of Sciences, URSS Series. Mathematics, 4, pp. 465-478.

[43] Maitra, A. P. and Sudderth, W. D. (2007) "Subgame-Perfect Equilibria in Stochastic Games," *Mathematics of Operations Research* 32, 711-722.

[44] Mertens, J.-F. and Parthasarathy, T. (1987) "Equilibria for Discounted Stochastic Games," CORE Discussion Paper 8750, Universite Catholique de Louvain.

[45] Mertens, J.-F. and Parthasarathy, T. (1991) "Nonzero-sum Stochastic Games," In T. E. S. Raghavan, et al. (eds.), *Stochastic Games and Related Topics in Honor of Professor L. S. Shapley*, Kluwer Academic: Dordrecht.

[46] Nowak, A. S. (1984) "On Zero-Sum Stochastic Games with General State Space I," *Probability and Mathematical Statistics* IV, 13-32.

[47] Nowak, A. S. (1985) "Existence of Equilibrium Stationary Strategies in Discounted Noncooperative Stochastic Games with uncountable State Space," *Journal Optimization Theory and Applications* 45, 591-602.

[48] Nowak, A. S. (1987) "Nonrandomized Strategy Equilibria in Noncooperative Stochastic Games with Additive Transitions and Reward Structure," *Journal Optimization Theory and Applications* 52, 429-441.

[49] Nowak, A. S. (2003) "On a New Class of Nonzero-Sum Discounted Stochastic Games Having Stationary Nash Equilibrium Points," *International Journal of Game Theory* 32, 121-132.

[50] Nowak, A. S. (2007) "On Stochastic Games in Economics," *Mathematical Methods of Operations Research* 66, 513-530.

[51] Nowak, A. S. and Raghavan, T. E. S. (1992) "Existence of Stationary Correlated Equilibria with Symmetric Information for Discounted Stochastic Games," *Mathematics of Operations Research* 17, 519-526.

[52] Parthasarathy, T. (1973) "Discounted, Positive and Noncooperative Stochastic Games," *International Journal of Game Theory* 2, 25-37.

[53] Parthasarathy, T. (1982) "Existence of Equilibrium Stationary Strategies in Discounted Stochastic Games," *Sankhya, Series A* 44, 114-127.

[54] Page, F. (1991) "Komlos Limits and Fatou's Lemma in Several Dimensions," *Canadian Mathematical Bulletin* 34, 109-112.

[55] Page, F. (2013) "Parameterized Games, Minimal Nash Correspondences, and Connectedness," SRC Discussion Paper 45, September, 2015.

[56] Pales, Z. and Zeidan, V. (1999) "Characterization of L^1-Closed Decomposable Sets in L^∞," *Journal of Mathematical Analysis and Applications* 238, 491-515.

[57] Rao, K. P. S. B. and Rao, M. B. (1972) "A Remark on Nonatomic Measures," *The Annals of Mathematical Statistics* 43, 369-370.

[58] Rieder, U. (1979) "Equilibrium Plans for Non-Zero Sum Markov Games," In O. Moeschlin and D. Pallaschke (eds.), *Game Theory and Related Topics*, North Holland: Amsterdam.

[59] Rogers, P. D. (1969) "Nonzero-Sum Stochastic Games," *Report ORC 69-8*, Operations Research Center, University of California, Berkeley.

[60] Rokhlin, V. A. (1949) "On the Principal Notions of Measure Theory," Matem. Sb., 25, 107-150.

[61] Salon, E. (1998) "Discounted Stochastic Games," *Mathematics of Operations Research* 23, 1010-1021.

[62] Shapley, L. (1953) "Stochastic Games," *Proceedings of the National Academy of Sciences USA* 39, 1095-1100.

[63] Sobel, M. (1971) "Noncooperative Stochastic Games," *Annals of Mathematical Statistics* 42, 1930-1935.

[64] Sobel, M. (1973) "Continuous Stochastic Games," *Journal of Applied Probability* 10, 597-604.

[65] Takahashi, M. (1964) "Equilibrium Points of Stochastic Noncooperative n-Person Games," *Journal of Science of the Hiroshima University Series A-I* 28, 95-99.

[66] van Mill, J. (2001) *The Infinite-Dimensional Topology of Function Spaces*, Elsevier, Amsterdam.

[67] Wagner, D. H. (1977) "Survey of Measurable Selection Theorems," *SIAM Journal of Control and Optimization* 15, 859-903.

[68] Whitt, W. (1980) "Representation and Approximation of Noncooperative Games," *SIAM Journal of Control and Optimization* 18, 33-48.

[69] Willard, S. (1970) *General Topology*, Addison-Wesley, Reading Massachusetts.

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