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JEL Classification: C14; G17; G32.

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This paper presents a methodology to infer multivariate densities that characterize the asset values for a system of financial institutions, and applies it to quantify systemic risk. These densities, which are inferred from partial information but are consistent with the observed probabilities of distress of financial institutions, outperform parametric distributions typically employed in risk measurement. The multivariate density approach allows us to propose complementary and statistically consistent metrics of systemic risk, which we estimate using market-based data to analyze the evolution of systemic risk in Europe and the U.S., throughout the financial crisis.

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1 Introduction

The global financial crisis demonstrated the speed and magnitude with which financial losses can propagate through financial systems. The crisis showed that initial losses in specific firms and markets could be magnified by contagion, leading to losses of calamitous proportions. Systemic risk —defined as “the risk of widespread disruption to the provision of financial services that is caused by an impairment of all or parts of the financial system, which can cause serious negative consequences for the real economy (IMF/FSB/BIS (2016); IMF (2013))”, is caused by externalities (direct exposure, fire sale pecuniary externalities, herding in the pricing of risk, etc.) that have the potential to amplify shocks up to the point of disrupting financial intermediation.

This paper considers financial systems as portfolios of entities and presents a methodology to infer the multivariate densities that characterize systems’ asset values. Data limitations remain an important constraint in the measurement of systemic risk. Given this constraint, our method offers important benefits. The densities are inferred from the limited data on individual financial entities that is usually readily available (equity prices and probabilities of default (PoD)). We show that the proposed distributions outperform the parametric distributions usually employed in risk measurement (Gaussian, t-distribution, mixture of normals) under the Diebold et al. (1998) Probability Integral Transformation criterion. The densities are then used to construct complementary measures of systemic risk that account for systems’ interconnectedness structures while also being able to incorporate changes in such structures when information changes. While a variety of complementary metrics of systemic risk can be constructed, these metrics are all consistent as they originate from a common multivariate density. Our method is easily implemented with publicly available market-based or supervisory data; hence, it can be used in a wide set of countries and financial stability metrics estimated can be updated easily and frequently.

Interconnectedness manifests itself through direct and indirect interlinkages across financial institutions (FIs) and markets. Direct interlinkages are mainly due to contractual obligations among financial entities. Indirect interlinkages can be caused by exposures to common risk factors, by asset fire sales (triggered by stressed entities) and asset sell offs (due to information asymmetries across agents). These interlinkages become particularly crucial in periods of high volatility, and can become self-reinforcing. Hence, interconnectedness is complex and likely unstable in periods of financial distress.

Given the importance of interconnectedness to the modeling of systemic risk, it is useful to think about the financial system as a portfolio of financial institutions, whose potential valuation can be represented by a multivariate density. Such a density characterizes (i) information of the individual firms' valuation in its marginal densities; and (ii) information of the function that describes the association across firms' valuation (or interconnectedness) in its copula function.¹ This twofold structure also results in two different information sets useful to policymakers. Micro-prudential surveillance would be interested in the marginals of this multivariate distribution, i.e. in identifying the risk that individual firms default; in the structural approach of Merton (1974) this is the risk that the value of the assets of the firm falls below a certain threshold (related to the capital buffer of the firm). Macro-prudential surveillance, on the other hand, would tend to focus on interconnectedness, especially in the tail of the marginal densities that characterize extreme asset values. Indeed, a key objective of policymakers is to assess the risk that the asset valuations of several financial firms simultaneously fall to levels low enough to provoke concurrent default.

A first challenge in characterizing such multivariate distributions is that our understanding of interconnectedness is usually limited, especially in the tail of the distribution. Frequently, what is available is (partial) information on individual firms; for example, information of individual firms' asset returns and in some cases, firms' likelihood of default. Obtaining information on the joint likelihood of default of financial institutions making up a financial system is usually very difficult.² Simultaneous defaults are infrequent and contribute little to the statistical relationships drawn from historical data. Moreover, financial systems have experienced significant structural changes, which make past relationships less reliable for modeling current interconnectedness. Ideally, models developed should thus be robust under such data restrictions. Although parametric assumptions may appear to resolve these issues, improper parametric calibration of risk models are known to lead to erroneous statistical inferences.³

¹In contrast to correlation, which only captures linear dependence, copula functions characterize linear and non-linear dependence structures embedded in multivariate densities

²While asset returns might allow to estimate return correlations across firms, such correlations represent linear dependence measures of "mean returns"; hence, do not capture adequately interconnectedness of extreme asset values (tail events).

³Koyluoglu et al. (2003) presents an interesting analysis of the consequences of the improper calibration of credit risk models.

A second challenge is that the measures of systemic risk obtained should be easily interpretable by policymakers, and therefore relate to the policymakers' policy reaction to systemic risk. Because policy reactions depend on the specific agencies interested in systemic risk (e.g. the central bank, the financial stability authority, the regulator), the current literature has provided a range of measures derived from Value-at-Risk, conditional probability, expected shortfalls, etc. (Bisias et al. (2012)). But the quantification of each measure is done using specific methodologies, which makes it difficult to ensure the metrics are consistent with each other.

This paper aims to address these two key challenges with the presentation of the Consistent Information Multivariate Density Optimization (CIMDO) methodology.⁴ CIMDO is a non-parametric procedure, based on the Kullback (1959) cross-entropy approach, to recover robust portfolio multivariate distributions from the incomplete set of information available for the modeling of systemic risk. In general, entropy approaches reverse the process of modeling data. Instead of assuming parametric probabilities to characterize the information contained in the data, these approaches use information in the data to infer unknown probability densities. In this specific case, the (unobserved) multivariate density characterizing the asset valuations and interconnectedness structure of a system of financial institutions is inferred from observed (but partial) information on the individual financial institutions in the system, i.e., their equity returns and probabilities of default (PoDs). These are observed or can be estimated from supervisory or market-based data. The CIMDO approach ensures that the inferred multivariate densities are consistent with the observed PoDs because the observed PoDs are used to impose restrictions on the moments of the multivariate density.

Using an extension of the Probability Integral Transform (PIT) criterion advocated by Diebold et al. (1998), this paper shows that CIMDO-inferred density forecasts perform better than parametric distributions forecasts, even when they are calibrated with the same information set. The CIMDO approach reduces the risk of density misspecification (especially in the tail of the distribution) because it recovers densities that are consistent with empirical observations of the PoDs.

As PoDs of individual financial institutions change across time, the CIMDO methodology allows to update, consistently with the changes in the PoDs, the resulting mul-

⁴The CIMDO was first introduced in an earlier version of this paper, Segoviano (2006).

tivariate densities and embedded copula functions. This is a key advantage over risk models that incorporate only linear dependence and assume it to be constant throughout economic cycles.⁵

To address the second challenge, we highlight that multivariate distributions inferred by the CIMDO methodology provide complementary financial stability measures that allow us to assess systemic risk from different perspectives, (i) tail risk, (ii) distress dependence, (iii) contagion losses, and (iv) contribution to systemic risk. Since these metrics are estimated as different moments of a common multivariate density, they provide different perspectives of systemic risk whilst being fully consistent. We also note our approach allows us to easily incorporate the risk contribution of non-banks (mutual funds, hedge funds, pension funds, etc.) into the analysis of systemic risk.

The paper's structure is as follows. Section 2 discusses the literature related to risk quantification and applications to systemic risk measurement. Section 3 introduces the CIMDO approach. Section 4 explains how the CIMDO dependence structure depends on information and assesses the sensitivity of CIMDO densities to misspecification. Section 5 evaluates the robustness of the CIMDO density under the Probability Integral Transform criterion proposed by Diebold et al. (1998). Section 6 proposes complementary financial stability measures that can be easily derived from CIMDO multivariate densities. The results of the application to the US and European banking and shadow-banking systems are discussed in Section 7, and Section 8 concludes on the benefits of the method, in particular for the calibration of theoretical models.

2 Literature

Systemic risk is caused by financial externalities that may be amplified by cyclical or structural vulnerabilities. A cyclical view of systemic risk indicates that during expansionary booms, funding constraints are looser and intermediaries can build up leverage and maturity mismatch. The greater risk appetite of intermediaries in boom times is reflected in higher asset valuations; hence intermediaries, in boom times, will tend to take more risk in the form of higher leverage and maturity transformation than is

⁵In comparison to traditional methodologies to model parametric copula functions, the CIMDO method avoids the difficulties of explicitly selecting a parametric form and calibrating its parameters. The approach allows to infer simultaneously with the CIMDO multivariate density the copula function that defines the interconnectedness structure across the marginal densities in the CIMDO multivariate distribution.

optimal from a social welfare perspective. In contrast, during economic contractions, evidence suggest that lenders become highly risk averse (Adrian et al. (2015)). Credit cycles implications for asset prices have been studied in the theoretical literature (Kiyotaki and Moore (1997)). This theoretical work has been further developed by assessing the interactions among the buildup of financial intermediary leverage, the implications for asset prices and the evolution of systemic tail risk (Adrian and Boyarchenko (2012); Gertler et al. (2012)).

Contagion among financial institutions can occur through direct linkages or through indirect links.⁶ Direct linkages include losses due to a counterparty's bankruptcy (Eisenberg and Noe (2001)) as well as funding shocks (Allen and Gale (2000); Freixas et al. (2000)). Indirect links can occur through a variety of channels, but the following channels have been those most discussed: (i) *fire sales and common exposures*, i.e. the sales of banks in distress affect asset prices, which can hurt other banks, especially in conjunction with collateral constraints (e.g. Bhattacharya and Gale (1987), Lorenzoni (2008); Stein (2012)); (ii) *information, when there is information asymmetry*: the information provided by the failure of a bank on the state of the economy can affect the valuation (and the probability of a bank run) for another bank (e.g. Garber and Grilli (1989)); (iii) *strategic complementarities*, for instance the failure of a bank can hamper the supply of funds and investment in the economy, reducing the profitability of the surviving banks (e.g. Acharya (2009)).

Methodologies that measure systemic risk have sought to capture the effect of linkages across financial entities in different ways. Bisias et al. (2012), provides a review of this empirical literature.⁷ The authors classify over thirty quantitative measures of systemic risk, within five categories ranging from probability distribution (statistical) measures to network analyses and macroeconomic measures. We briefly explain here the main probability distribution measures to focus on their similarities and differences with the CIMDO-based financial stability measures presented in Section 6.

Statistical measures construct estimates of correlations, of probabilities, or of conditional losses, for events of joint distress. These measures are not structural and most often cannot attribute true causality (although Granger-causality is sometimes used), which has the advantage that the measures are informative independently of theoretic-

⁶See De Bandt and Hartmann (2000) for a more detailed survey of the literature

⁷Acharya et al. (2017) provides another, shorter, survey.

cal priors. They also capture both direct and indirect linkages. A limitation, common to all these measures, is that they cannot provide information on the channels of contagion since they are *reduced-form*.

The CoVaR model of Adrian and Brunnermeier (2016) estimates the Value at Risk (VaR) of a firm, conditional on another firm being in distress. The CoVaR can be estimated with quantile regressions, using the time variation to capture comovement. Quantile regressions allow a better fit of the model in the lower tail of the distribution (domain of distressed values) that the user is interested in.⁸ The Co-Risk measure of IMF (2009) is similar in spirit to CoVaR, except that Co-Risk examines the CDS spread of one firm (as opposed to the asset value in CoVaR), conditional on the CDS spread of another firm, each at the respective 95th percentile of its empirical distribution. However, nothing guarantees that the PoDs predicted by Co-Risk are *consistent* with the PoDs that are empirically observed. The conditional probability measures we propose are on the contrary consistent with the observed PoDs. In addition, the multivariate density incorporates the complete interconnectedness structure, and thus the financial stability measures we can construct are not limited to pairwise conditional probabilities.

Huang et al. (2009) and Huang et al. (2012) proposed a measure of systemic risk (the Distressed Insurance Premium, DIP) based on the calculation of a hypothetical, forward-looking, insurance premium against large losses suffered by a system of financial firms. The method primarily relies on the construction of high-frequency correlations of asset returns for the financial institutions analyzed. The individual institutions' PoDs are deduced from CDS spreads and a standard portfolio credit risk model (Hull and White (2004); Tarashev and Zhu (2008)) is used to estimate the expectation of portfolio credit losses. The indicator of financial stability is thus fundamentally based on the (parametric) assumption that asset returns are distributed as multivariate log-normal. This is a key limitation that the CIMDO approach addresses.

Acharya et al. (2017) show that a firm's contribution to systemic risk can be captured by its systemic expected shortfall (SES), which is the probability of a systemic crisis multiplied by the loss of the firm conditional on such a crisis. SES is a well-

⁸With the use of rolling regressions, it is also possible to estimate backward-looking time-varying CoVaR. Adrian and Brunnermeier (2011) also propose a forward-looking estimate of CoVaR, but this measure is constructed indirectly: first, a regression of backward-looking CoVaRs on structural firm characteristics is estimated to identify good predictors of future CoVaR. Second, this model is applied to current data to predict CoVaR.

defined variable that has important theoretical implications —SES is a key component of the optimal systemic risk tax — but it is also a variable that can be estimated as a linear combination of the marginal expected shortfall (measured as the 5th percent worst equity returns), of leverage, of excess returns on bonds due to credit risk, and the excess costs of financial distress. The authors also link the SES to the capital increase that regulators recommended following the US banking sector stress tests of February 2009.

Finally, Diebold and Yilmaz (2009) and Diebold and Yilmaz (2014) have suggested measures of interconnectedness based on weighted, directed networks, using VAR forecast error variance decompositions to estimate the network’s weighted adjacency matrix.⁹ An issue with such measures is that they are difficult to convert to probability or to monetary units, which are most valuable for policymaking.

3 Consistent Information Multivariate Density Optimisation (CIMDO)

In order to account for the potential loss propagation of financial entities when measuring systemic risk, it is useful to think about the financial system as a portfolio of financial entities, whose potential individual values can be represented by a multivariate density. The *structural approach* of Merton (1974) is then the starting point to model default risk. The premise of the structural approach is that a firm’s underlying asset value evolves stochastically over time, and that default is triggered by a drop in the firm’s asset value below a pre-specified barrier, henceforth called the default-threshold, which is modeled as a function of the firm’s leverage structure.

The difficulty in extending the model to a system of firms comes from the choice and calibration of a multivariate distribution. Because financial assets’ returns exhibit heavy tails, Glasserman et al. (2002) proposed a multivariate distribution where marginals follow t -distributions with the same degrees of freedom; however, such a framework is not sufficiently flexible to account for risk heterogeneity among financial

⁹Diebold and Yilmaz (2014) also noted that Acharya et al. (2017)’s Marginal Expected Shortfall and Adrian and Brunnermeier (2016)’s CoVaR were specific measures based on aggregations of a weighted directed network.

institutions.¹⁰ Mixture models (McLachlan and Basford (1988) and Zangari (1996)) provide an alternative option but their calibration is also difficult.¹¹ Copula functions, which allow modelers to account for linear and non-linear dependence structures have also been used (see Gagliardini and Gouriroux (2003); Schnbucher (2003); Embrechts et al. (2003)). Copula modeling is a step in the right direction but it has shortcomings common to parametric modeling, in particular the need to choose a specification and calibration of (parametric) copula functions, and the need to calibrate dependence — often using a time invariant parameter.

This paper proposes the CIMDO approach, based on Kullback (1959)’s cross-entropy approach, to recover multivariate distributions from the incomplete set of information available for the modeling of systemic risk. The starting point of this literature is Shannon (1948), who defined a unique function that measures the uncertainty of a collection of events (entropy). Jaynes (1957) proposed to make use of this entropy concept to choose an unknown distribution of probabilities when only partial information is available. Kullback (1959) and Good (1963) extended the proposal to cases where, in addition to moment constraints, some form of conceptual knowledge exists about the properties of the system that can be expressed in the form of a prior probability distribution (Golan et al. (1996)).

Following this literature, we propose to infer the unknown multivariate distribution that characterizes the implied asset values of a portfolio of firms from the observed PoDs of the firms making up the portfolio and from a prior multivariate distribution. The cross-entropy approach recovers the distribution that is closest to the prior distribution but that is consistent with the PoDs, which are empirically observed.

¹⁰Extensions of multivariate t -distributions that allow for different degrees of freedom in their marginals are possible, but under these assumptions, the multivariate t -distributions are not fully described by their variance-covariance matrices.

¹¹Mixture models assume that the firm’s logarithmic asset values are generated from a mixture of two different normal distributions: the distribution of the *quiet* state and the distribution of the *volatile* state, which has a certain probability of occurrence. An attractive property of the mixture model is that its distribution exhibits heavy tails due to the random nature of volatility. In the univariate case, it is necessary to estimate five parameters (two variances, two means and the probability of being in a volatile state). In the multivariate case calibration becomes even more difficult, as it is necessary to calibrate two covariance matrices corresponding to the *quiet* and *volatile* states for the multivariate distributions.

3.1 Objective function and priors

For a portfolio, containing assets given to M different risk, whose logarithmic returns are characterized by the random variables l^1, \dots, l^M , finding a multivariate distribution $p(l^1, \dots, l^M)$ consistent with a set of observations is equivalent to solving the constrained minimization problem

$$\min_{p(\dots) \in S} C [p(l^1, \dots, l^M), q(l^1, \dots, l^M)] = \int_{l^M} \dots \int_{l^1} p(l^1, \dots, l^M) \ln \left[\frac{p(l^1, \dots, l^M)}{q(l^1, \dots, l^M)} \right] dl^1 \dots dl^M$$

where the set of constraints S (described below) is the set of conditions given by the available information (for instance the unconditional probabilities of default) and the condition that the posterior probability distribution sums to 1. In the interest of parsimony, the simpler bivariate problem ($M = 2$) is presented, although all the results are directly applicable when $M > 2$. The two assets are characterized by their logarithmic asset returns x and y and the minimization problem is simply defined as $\min_{p(\dots) \in S} C [p(x, y), q(x, y)] = \int \int p(x, y) \ln \left[\frac{p(x, y)}{q(x, y)} \right] dx dy$, where $q(x, y) \in \mathbb{R}^2$ is the prior distribution and $p(x, y) \in \mathbb{R}^2$ the posterior distribution. The Kullback (1959) cross-entropy criteria $C [p(x, y), q(x, y)]$ can be thought of as the weighted average (with weights $p(x, y)$) of the relative distance between p and q ($\ln \left[\frac{p(x, y)}{q(x, y)} \right]$) and is a measure of distance between the prior distribution q and the posterior distribution p .¹²

The objective of the minimization problem is therefore to choose the posterior distribution p that is closest to the prior and consistent with the constraints S . The prior distribution q can be chosen differently depending on the problem at hand but can either represent uninformative priors, be calibrated using theoretical priors and economic intuition, or consistently with some simple empirical observations.¹³

3.2 Moment-consistency constraints

The information provided by the probabilities of default of each type of asset is incorporated in a set of moment-consistency constraints that modify the shape of the posterior multivariate distribution. The moment-consistency constraints are restrictions on the

¹²The Kullback divergence is not a distance metric though. In particular it is not symmetric and does not satisfy the triangle inequality.

¹³In the application to a portfolio of banks (section 7) the third option was chosen and q was calibrated as a multivariate normal distribution with the correlation matrix equal to the correlation of equity returns, computed in centered rolling windows.

marginals of the portfolio multivariate distribution. Imposing these constraints on the optimization problem guarantees that the posterior multivariate distribution contains marginal densities that sum to the observed PoDs in the region of default¹⁴ (we use the convention that the zone of default is $[X_d^m, \infty]$ for $m \in \{x, y\}$, i.e. $-x$ and $-y$ represent the equity returns):

$$\int \int p(x, y) \chi_{[x_d^x, \infty]} dx dy = PoD_t^x \text{ and } \int \int p(x, y) \chi_{[x_d^y, \infty]} dy dx = PoD_t^y \quad (1)$$

$p(x, y)$ is the *posterior* multivariate distribution that represents the unknown to be solved. In addition, probabilities must be positive and sum to 1.

3.3 Solution

Let us define the functional

$$\mathcal{L}(x, y, p, \lambda) = \ell(x, y, p) + \lambda_x \varphi_1(x, y, p) + \lambda_y \varphi_2(x, y, p) + \mu \varphi_3(x, y, p)$$

where $\lambda_x, \lambda_y, \mu$ are lagrange multipliers, $\ell(x, y, p) = p(x, y) [\ln p(x, y) - \ln q(x, y)]$ is the cost function and, $\varphi_1(x, y, p) = p(x, y) \chi_{[x_d^x, \infty]}$, $\varphi_2(x, y, p) = p(x, y) \chi_{[x_d^y, \infty]}$, $\varphi_3(x, y, p) = p(x, y)$ are the functionals associated to the moment-consistency constraints in (1). Using the calculus of variations, there exist lagrange multipliers $\lambda_x, \lambda_y, \mu$, such that the solution \hat{p} satisfies the Euler-Lagrange equation $\frac{d\mathcal{L}(\hat{p})}{dp} = 0$, which is:

$$\hat{p}(x, y) \left[\frac{1}{\hat{p}(x, y)} \right] + [\ln \hat{p}(x, y) - \ln q(x, y)] + \lambda_x \chi_{[x_d^x, \infty]} + \lambda_y \chi_{[x_d^y, \infty]} + \mu = 0$$

The *posterior* multivariate density is the solution of this problem (Golan et al. (1996) show the solution is unique):

$$\hat{p}(x, y) = q(x, y) \exp \left\{ - \left[1 + \mu + \left(\lambda_x \chi_{[x_d^x, \infty]} \right) + \left(\lambda_y \chi_{[x_d^y, \infty]} \right) \right] \right\} \quad (2)$$

where μ, λ_x and λ_y are solutions of the system

¹⁴The region of default defines the set of events under which the firm is considered to be in default, and the concept of default used should be consistent with the definition of default to which the observed PoDs refer. In this paper, the concept of default, or rather of distress, is broader than that of default in the Merton model because the CDS spreads used to measure default risk are not narrowly based on default events. Thus, the region of default (or rather, distress) does not correspond to the risk that equity falls to 0.

$$\begin{cases} \int \int \hat{p}(x, y) \chi_{[x_d^x, \infty]} dx dy = PoD_t^x \\ \int \int \hat{p}(x, y) \chi_{[x_d^y, \infty]} dy dx = PoD_t^y \\ \int \int \hat{p}(x, y) dx dy = 1 \end{cases} \quad (3)$$

3.4 Data requirements

The data required for CIMDO is: (i) data to calibrate a prior density; (ii) probabilities of default of individual firms; and (iii) thresholds in the value of assets that define the zone of default. The prior can be calibrated using any relevant information, for instance on asset or equity returns, using stock market data to calibrate e.g. a normal distribution or or t -distribution.¹⁵ The observed PoDs are crucial inputs to CIMDO. These can be obtained from bond prices or CDS spreads (assuming a certain recovery rate and price of risk), from a Merton model's assessment of default frequency, or from commercial databases (e.g. Moody's KMV EDF).

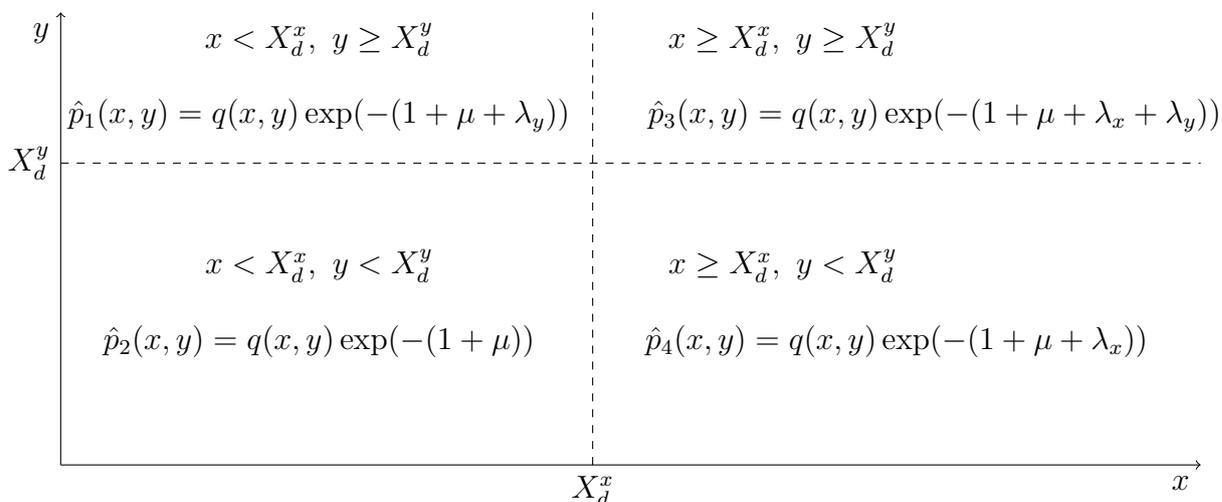
Finally, for each firm, the region of default needs to be fixed by calibrating a threshold (i.e. X_d^x and X_d^y) so that changes in PoD_t^x and in PoD_t^y affect the shape of the posterior distribution rather than the thresholds themselves. The default-threshold is fixed to an average (through time) that is consistent with the historical average of the probability of default for each asset, \overline{PoD}^m , $m \in \{x, y\}$, and with the prior distribution. For instance, if the prior distribution is a bivariate t -distribution, the historical average of the default threshold for each borrower is set to $X_d^x = \tau^{-1}(\overline{\alpha}^x)$ and $X_d^y = \tau^{-1}(\overline{\alpha}^y)$, where $\tau(\cdot)$ is the distribution *cdf* and $\overline{\alpha}^x = 1 - \overline{PoD}^x$ and $\overline{\alpha}^y = 1 - \overline{PoD}^y$ (with the model conventions, the region of default for each obligor is described in the upper part of a distribution). Given these inputs, at each time t the solution of system (3) is found to be the three scalars λ_x, λ_y and μ , which are used in conjunction with the prior density q to obtain the posterior CIMDO density \hat{p} according to equation (2).

4 How CIMDO incorporates interconnectedness structures

CIMDO provides a simple way to adjust prior distributions to available information. The adjustment is flexible since it varies depending on the domain. Figure

¹⁵We discuss in section 4 the robustness of CIMDO to mis-specifications in the prior.

Figure 1: CIMDO-density, Adjustment factor



1 shows how the adjustment between the prior and the posterior, $\exp\{-[1 + \mu + (\lambda_x \chi_{[X_d^x, \infty)}) + (\lambda_y \chi_{[X_d^y, \infty)})]\}$, depends on the domain, even though only three parameters, μ , λ_x and λ_y need to be computed.¹⁶ Moreover, Figure 1 shows that when $\lambda_x < 0$ and $\lambda_y < 0$, which happens when the PoDs implied by the prior are below the observed PoDs, the adjustment in the zone of joint default (top-right corner, captured by \hat{p}_3) is $\exp(-\mu) \exp(-\lambda_x) \exp(-\lambda_y)$ and is thus higher than the adjustment applied in the zones of single default (top left corner, captured by \hat{p}_1 , or bottom right corner, captured by p_4), i.e. $\exp(-\mu) \exp(-\lambda_y)$ or $\exp(-\mu) \exp(-\lambda_x)$. Thus, CIMDO strengthens dependence when marginal PoDs are underestimated by the prior.

The following propositions provide additional hindsights into how CIMDO modifies densities, in particular in relation to the modeling of dependence. Proposition 1 shows how the copula of the prior density is modified by CIMDO. In particular, it shows how the dependence structure is a function of the lagrange multipliers $\lambda_x, \lambda_y, \mu$. Proposition 2 shows how the lagrange multipliers depend on the PoDs implied by the prior and on the PoDs that are used as constraints to the minimization problem. Together, Proposition 1 and Proposition 2 thus show how the dependence structure is a function of the prior PoDs and of the observed PoDs.

¹⁶This is possible of course because the thresholds are fixed. If the thresholds were not fixed, the model would be under-identified.

Finally, Proposition 3 shows, using a t -student distribution as an example, that the adjustment provided by CIMDO is not sensitive to the correlation of the prior when $X_d^x, X_d^y \rightarrow +\infty$ or when $\lambda_x, \lambda_y \rightarrow 0$. This is important because it implies that CIMDO is *robust* to a mis-specification of the prior correlation if default probabilities are small or if the prior is nearly consistent with the observed PoDs.

Proposition 1. *CIMDO-copula*

Assume the prior density is $q(x, y)$. The copula of q is $c_q(u, v) = \frac{q[F^{-1}(u), H^{-1}(v)]}{f[F^{-1}(u)]h[H^{-1}(v)]}$, where u, v are the marginal cdf F and H of q , i.e. $u = F(x) = \int_{-\infty}^x \int_{-\infty}^{\infty} q(x, y) dy dx$ and $v = H(y) = \int_{-\infty}^y \int_{-\infty}^{\infty} q(x, y) dx dy$, and where the marginal densities are $f(x) = \int_{-\infty}^{\infty} q(x, y) dy$ and $g(y) = \int_{-\infty}^{\infty} q(x, y) dx$. Then, the dependence structure of CIMDO can be represented by the following CIMDO-copula function:

$$c_c(u, v) = \frac{q[F_c^{-1}(u), H_c^{-1}(v)] \exp\{-[1 + \mu]\}}{\int_{-\infty}^{+\infty} q[F_c^{-1}(u), y] \exp\{-\lambda_x \chi_{[X_d^x, \infty)}\} dy \int_{-\infty}^{+\infty} q[x, H_c^{-1}(v)] \exp\{-\lambda_y \chi_{[X_d^y, \infty)}\} dx}$$

where $u = F_c(x)$, $v = H_c(y)$, and the marginal densities are

$$f_c(x) = \int_{-\infty}^{\infty} q(x, y) \exp\{-[1 + \mu + (\lambda_x \chi_{[X_d^x, \infty)} + (\lambda_y \chi_{[X_d^y, \infty)})]\} dy$$

$$h_c(y) = \int_{-\infty}^{\infty} q(x, y) \exp\{-[1 + \mu + (\lambda_x \chi_{[X_d^x, \infty)} + (\lambda_y \chi_{[X_d^y, \infty)})]\} dx$$

Proof. By using the marginal densities f_c and h_c in the definition of a copula. □

Proposition 2. *Modeling of dependence*

Assume that $(\mu, \lambda_x, \lambda_y)$ solve the system (3) taking into account two probabilities of default PoD_x and PoD_y , with a iid prior distribution q that does not embed prior dependence (i.e. $q(x, y) = q(x)q(y)$). In addition, assume $(\hat{\mu}_1, \hat{\lambda}_x)$ is the CIMDO solution for a univariate problem, taking into account the information PoD_x only, and that $(\hat{\mu}_y, \hat{\lambda}_y)$ is the CIMDO solution for the univariate problem taking into account PoD_y only. Define $Q_i, i \in \{x, \bar{x}, y, \bar{y}, xy, \bar{x}\bar{y}, x\bar{y}, \bar{y}\bar{y}\}$ as the different probabilities of default (index without a bar) or non-default (index with a bar) under the prior distribution

(see also Appendix). Then, the approximations

$$\lambda_x \approx \frac{\hat{\lambda}_x + \hat{\lambda}_y \frac{Q_x Q_y - Q_{xy}}{Q_x Q_{\bar{x}}}}{1 - \frac{(Q_x Q_y - Q_{xy})^2}{Q_x Q_{\bar{x}} Q_y Q_{\bar{y}}}}; \quad \lambda_y \approx \frac{\hat{\lambda}_y + \hat{\lambda}_x \frac{Q_x Q_y - Q_{xy}}{Q_y Q_{\bar{y}}}}{1 - \frac{(Q_x Q_y - Q_{xy})^2}{Q_x Q_{\bar{x}} Q_y Q_{\bar{y}}}} \quad (4)$$

show that:

- i) the adjustment to the prior multivariate density (captured by the lagrange multipliers λ_x, λ_y) differs from the adjustment for the univariate densities ($\hat{\lambda}_x, \hat{\lambda}_y$);
- ii) λ_y is a function of both PoD^y (as reflected in $\hat{\lambda}_y$) and of PoD^x (as reflected in $\hat{\lambda}_x$).¹⁷
- iii) when the prior assumes the distress events are independent (i.e. $Q_{xy} = Q_x Q_y$), $\lambda_x \approx \hat{\lambda}_x$ and $\lambda_y \approx \hat{\lambda}_y$. CIMDO does not create a “spurious” dependence structure if it was not embedded in the prior.

Proof. See Appendix □

Proposition 3. *Sensitivity to the correlation in the prior*

Assume the prior is a centered bivariate t-distribution, with ν degrees of freedom and correlation coefficient σ . Define $J = \frac{\nu^{\nu/2}}{2\pi}(\nu + X_d^{x2} + X_d^{y2})^{-\nu/2}$, and define $\tilde{Q}_i, i \in \{xy, \bar{x}y, x\bar{y}, \bar{y}\bar{y}\}$ as the prior joint probabilities of default/non-default if the prior was distributed with a correlation coefficient of 0 (see also Appendix). Then the following approximations

$$\begin{aligned} \lambda_x &= -\ln(PoD_x) - 1 - \mu + \ln(\tilde{Q}_{xy}e^{-\lambda_y} + \tilde{Q}_{x\bar{y}} + (e^{-\lambda_y} - 1)J\sigma + \mathcal{O}(\sigma^2)) \\ \lambda_y &= -\ln(PoD_y) - 1 - \mu + \ln(\tilde{Q}_{xy}e^{-\lambda_x} + \tilde{Q}_{\bar{x}y} + (e^{-\lambda_x} - 1)J\sigma + \mathcal{O}(\sigma^2)) \\ \mu &= -1 + \ln[\tilde{Q}_{xy}e^{-\lambda_x}e^{-\lambda_y} + \tilde{Q}_{\bar{x}y}^{-\lambda_y} + \tilde{Q}_{x\bar{y}}e^{-\lambda_x} + \tilde{Q}_{\bar{y}\bar{y}} \\ &\quad + (e^{-\lambda_x}e^{-\lambda_y} - e^{-\lambda_y} - e^{-\lambda_x} + 1)J\sigma + \mathcal{O}(\sigma^2)]. \end{aligned} \quad (5)$$

show that:

- i) the lagrange multipliers depend on the prior’s correlation coefficient σ , but:

¹⁷The result is symmetric for λ_x .

- ii) when $\lambda_x, \lambda_y \rightarrow 0$, the adjustment factor due to CIMDO is insensitive to the correlation coefficient σ ;
- iii) when $X_d^x, X_d^y \rightarrow +\infty$, the adjustment factor due to CIMDO is insensitive to the correlation coefficient σ .

Proof. See Appendix □

5 Density evaluation

Do densities derived with CIMDO improve upon the performance of standard parametric models, even when these models are calibrated well enough to be consistent with the observed data? This section conducts an evaluation of density forecasts using Diebold et al. (1998)'s Probability Integral Transform (PIT) method. Density evaluation is a complex problem because it is impossible to rank two incorrect density forecasts such that all users agree with the ranking. Ranking depends on the specific loss functions of the users.¹⁸ However, Diebold et al. (1998) noted that “if a forecast coincides with a random variable true data-generating-process (DGP), then it will be preferred by all forecast users, regardless of loss function”. Although determining whether a forecast equals the true DGP is difficult because the true DGP is never observed, Diebold et al. (1998) propose a method based on the Rosenblatt (1952) Probability Integral Transform (PIT) that assesses whether the realized PIT's of the forecast densities are distributed **iid** $\mathbf{U}(\mathbf{0}, \mathbf{1})$.

5.1 Theory

Diebold et al. (1999) also extend this method to the M -multivariate case, when there are T time-series observations of the realized process. They factorize each period's t , joint forecast density into the product of their conditionals:¹⁹

$$p_{t-1} = (l_t^1, \dots, l_t^M) = p_{t-1}(l_t^M / l_t^{M-1}, \dots, l_t^1) \dots p_{t-1}(l_t^2 / l_t^1) \cdot p_{t-1}(l_t^1) \quad (6)$$

This procedure produces a set of $M - 1$ conditionals and 1 marginal density. The PIT's of the l^m random variable realizations under these M series will be **iid** $\mathbf{U}(\mathbf{0}, \mathbf{1})$,

¹⁸Diebold et al. (1998) note that “the result is analogous to Arrow's impossibility theorem. The ranking effectively reflects a social welfare function, which does not exist.”

¹⁹Note that the $M -$ multivariate density can be factorized into $M!$ ways at each period of time t .

individually and also when taken as a whole, if the multivariate density forecasts are correct (Diebold et al. (1998)). We propose a variant of this test because CIMDO recovers densities using only information at each period of time t , and thus we want to evaluate the density forecasts using a cross section of realizations, as opposed to a time series. The test thus does not use any information ‘along time’, it only uses cross-sectional information at a given time. The test is presented for two assets, but the extension to more assets is trivial.

Proposition 4. *Probability Integral Transform*

Two assets have logarithmic returns x and y , with bivariate density $p(x, y)$. Define the Probability Integral Transform under the distribution f as $P(x) = \int_{-\infty}^x f(t)dt$. Then, define u and v as

$$\begin{aligned} u &= P(x) \iff x = P^{(-1)}(u) \\ v &= P(y|x) \iff y = P^{(-1)}(v|x) \end{aligned}$$

u, v are always independent. In addition, if f is the true distribution (i.e. if $f = p$), then u, v are distributed $\mathbf{U}(\mathbf{0}, \mathbf{1})$.

Proof. See Appendix □

In time series settings, empirical tests are that $(u, v) \sim \mathbf{iid} \mathbf{U}(\mathbf{0}, \mathbf{1})$. In our case, independence of the conditionals and the marginals is proven and it is not necessary to test for it. The only test needed is that u and v are uniformly distributed over $[0, 1]$.

We run 10,000 Monte Carlo simulations in order to perform the density evaluation. Density evaluation requires the following steps:

- i) Assume the DGP is a multivariate t -distribution with non-zero mean, with identity scale matrix, and 6 degrees of freedom²⁰ to match two PoDs (we choose $PoD^x = 0.22$ and $PoD^y = 0.29$). The location of the DGP is thus $[0.3613, 0.4004]$.
- ii) Calibrate a multivariate *centered* normal (referred later as NCon), a multivariate *centered* t -distribution (TCon) and a mixture of normals (NMix). The calibration ensures that the PoDs of the assumed parametric densities are consistent with the empirically observed PoDs. However, even if the shape of the distribution is

²⁰Empirical evidence presented in Hansen (1994) and in Bekaert and Harvey (2003) indicate that this is a reasonable assumption.

known (a t -distribution with 6 degrees of freedom), the problem of calibrating a t -distribution with two PoDs is under-identified. The choice to calibrate the means at 0 implies that TCon (and a fortiori NCon) is not identical to the DGP.

- iii) Infer the CIMDO-density, using a standard normal distribution as a prior (which also has the wrong location), and the empirically observed PoDs.
- iv) Decompose the competing distributions into the product of their marginal and conditional probabilities, as indicated in equation (6).
- v) Compute the PITs of the random variable realizations under the distributions $z_{x|y} = P(x|y)$, $z_y = P(y)$, $z_{y|x} = P(y|x)$, $z_x = P(x)$, where P represents the cdf of each of the evaluated distributions.
- vi) Test whether the series $z_{x|y}$, z_y are iid $U(0, 1)$.²¹ The test only involves a test of uniformity (Proposition 4) performed both thanks to the Kolmogorov-Smirnov (K-S) test,²² and with a simple plot of the z -variables' cdf along the 45 degree line. In particular, the focus will be on the region near default, where the decision-maker's losses due to an imperfect forecast would arguably be the largest.

5.2 Results

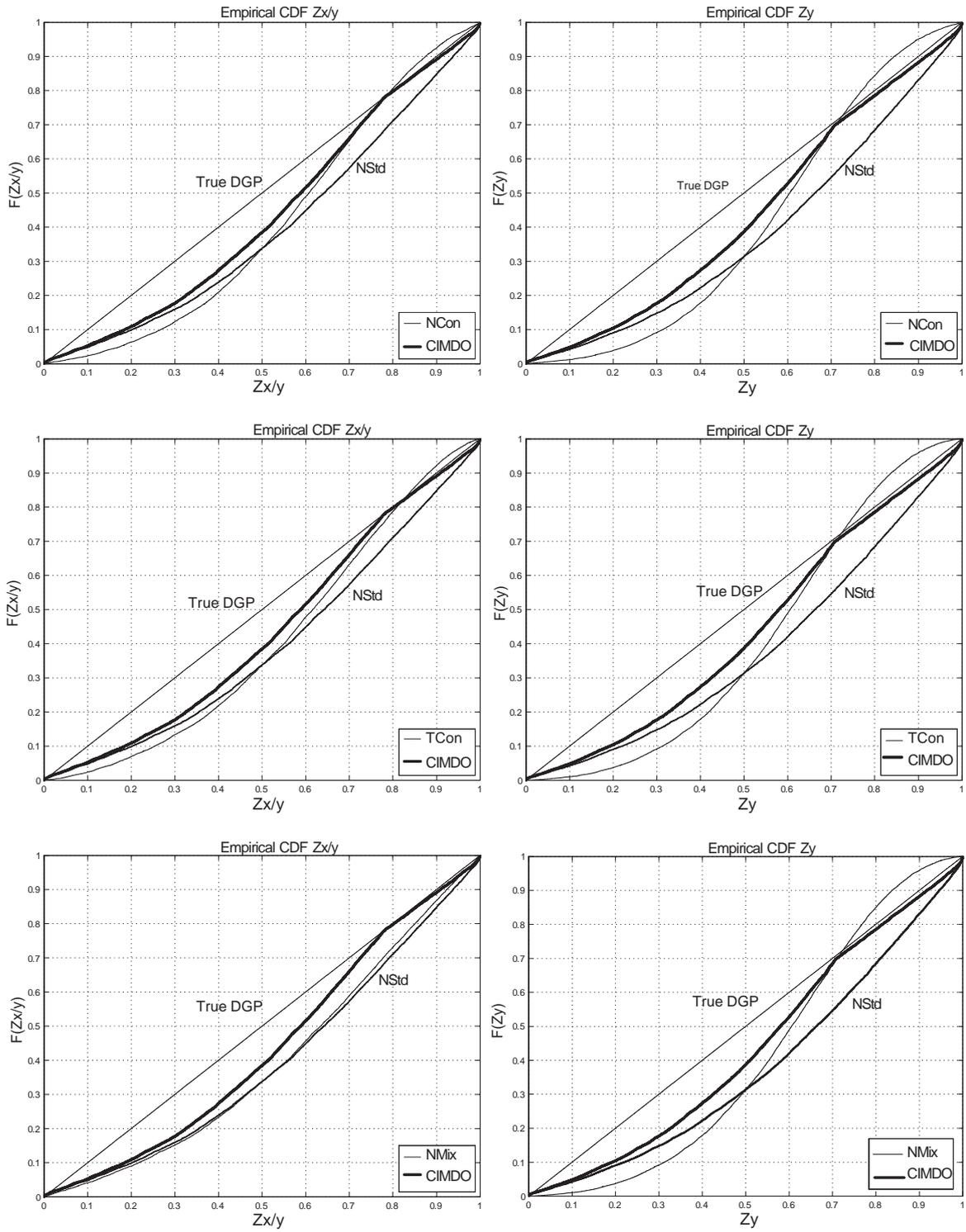
The cdfs of the different PITs are presented in Figure 2. The cdfs of the $z_{x|y}$ series are shown in the first column of charts, and the cdfs of the z_y series are shown in the second column of charts. In each chart, the cdf derived from the CIMDO density is plotted along the 45 degree line (cdf of the PIT of the 'true' DGP), along the cdf of a standard normal distribution (labelled Nstd; this is a naive, non-calibrated density), and along the density of either (i) the calibrated multivariate normal (NCon, top row charts); (ii) the calibrated multivariate t -distribution (TCon, middle row charts); (iii) the mixture of normal model (NMix, bottom row charts).

Since the CIMDO PIT's cdf is always closer to the DGP than the standard normal distribution's cdf, CIMDO outperforms the standard normal distribution under the PIT criterion. This is not surprising, since a standard normal distribution was a naive calibration, inconsistent with the empirical facts. However, the PIT of the standard normal distribution gives an idea of the degree of misspecification that can

²¹We also tested for the normality of the series $z_{y|x}$, z_x . The results are similar and not presented.

²² $H_0 : F = U(0, 1)$, $H_a, F \neq U(0, 1)$

Figure 2: Probability Integral Transform



Source: authors' calculations.

Table 1: Kolmogorov-Smirnov Tests

K-S Test: $z_{x/y}$					
	CIMDO	NStd	NCon	TCon	NMix
K-Statistic	0.1296	0.1654	0.1932	0.1834	0.1700
Critical Value	0.0136	0.0136	0.0136	0.0136	0.0136

K-S Test: z_y					
	CIMDO	NStd	NCon	TCon	NMix
K-Statistic	0.1287	0.1883	0.2251	0.2237	0.2218
Critical Value	0.0136	0.0136	0.0136	0.0136	0.0136

Source: authors' calculations.

be reached. More importantly, the CIMDO distribution outperforms all the competing distributions, especially in the region of default (upper right corner of each chart), even though these distributions were calibrated to match the same observed PoDs. This result shows that CIMDO uses restricted information in a more efficient manner. Overall, whilst the fit outside the region of default is not as good as in the region of default (the null hypothesis of the Kolmogorov-Smirnov test is always rejected – see Table 1), CIMDO densities outperform the competing distributions, especially in the region of default.

6 Financial Stability Measures

Given a multivariate density of asset returns for a system of firms, it is possible to propose a variety of financial stability measures that can be updated daily. Although these measures are consistent with each other, since they are all derived from the same underlying multivariate density of asset values, the different measures correspond to different views of what systemic risk can mean. This is especially useful because different agencies (the monetary authority, the regulator, the Treasury) tend to consider systemic risk from different angles.

6.1 Measures of tail risk

Even if financial stability were not an independent objective, an inflation targeting central bank would need to care about financial stability because of its impact on output and inflation. Then, according to Woodford (2012) “the question of greatest

concern is [] the probability of a bad joint outcome”. This view leads to a first proposed measure of systemic risk, the probability that all the financial institutions in a given system be in distress at the same time (the *Joint Probability of Distress*, JPoD). For simplicity of presentation, the JPoD formula is shown for a financial system made of three firms with asset returns x_1, x_2, x_3 :

$$JPoD = \int_{X_d^{x_3}}^{\infty} \int_{X_d^{x_2}}^{\infty} \int_{X_d^{x_1}}^{\infty} \hat{p}(x_1, x_2, x_3) dx_1 dx_2 dx_3 \quad (7)$$

We also compute a *Financial Stability Index* (FSI) as the expected number of banks becoming distressed given that at least one bank has become distressed.²³ For example, for a system of two banks, the FSI is defined as

$$FSI = (P(x_1 \geq X_d^{x_1}) + P(x_2 \geq X_d^{x_2})) / (1 - P(x_1 < X_d^{x_1}, x_2 < X_d^{x_2})) \quad (8)$$

and the different probabilities are computed by numerical integration of the multivariate density.

6.2 Measures of dependence

Even if systemic risk were not affecting the path of output and inflation, the externalities in financial intermediation could require corrective regulation. The *Distress Dependence Matrix* (DiDe) provides measures of inward and outwards linkages. It is defined as the matrix of the probability of distress of the firm specified in the row, given that the firm specified in the column becomes distressed:

$$(DiDe)_{i,j} = P(x_i \geq X_d^{x_i} \mid x_j \geq X_d^{x_j}) \quad (9)$$

Although conditional probabilities do not imply causation, this set of pairwise conditional probabilities can provide important insights into interlinkages and the likelihood of contagion between the firms in the system.

An extension of the DiDe is the *Probability of Cascade Effects* (PCE), i.e. the likelihood that one, two, or more institutions, become distressed given that a specific firm

²³See also Huang (1992) and Hartmann et al. (2004). Huang (1992) shows that this measure can also be interpreted as a relative measure of banking linkage. When $FSI \rightarrow 1$, banking linkage is weak (asymptotic independence). As the value of the FSI increases, banking linkage increases (asymptotic dependence).

becomes distressed. This measure quantifies the potential domino effects of a firm and is thus an indicator of its systemic importance. For example, in a financial system with four firms, where the events x_1^d , x_2^d , x_3^d , and x_4^d refer to the distress events, the PCE given that firm 1 becomes distressed, is defined as follows:

$$PCE_1 = P(x_2^d|x_1^d) + P(x_3^d|x_1^d) + P(x_4^d|x_1^d) - [P(x_2^d, x_4^d|x_1^d) + P(x_2^d, x_3^d|x_1^d) + P(x_3^d, x_4^d|x_1^d)] + P(x_2^d, x_3^d, x_4^d|x_1^d) \quad (10)$$

6.3 Measures of expected losses

One of the most salient consequence of systemic risk is the cost to taxpayers that financial support policies can require if a crisis materializes. Laeven and Valencia (2013), in their study of the 147 banking crises that affected 116 countries over the period 1970-2011, find that the fiscal costs of financial support policies averaged 7 percent of GDP of the crisis country, and reached more than 40 percent of GDP in several occasions. When government agencies face a financial crisis, the issue of whether to intervene to stop a contagion involves a trade-off between the immediate costs of support policy and the potential future costs if contagion is not halted.

Measures of expected shortfall help inform this tradeoff. We define the financial system's Systemic Expected Shortfall as the equity losses of the portfolio of financial firms given that the portfolio is performing below its q percentile. For a system made of three financial institutions $S = \{1, 2, 3\}$, the Systemic Expected Shortfall is

$$\mathcal{V}(S) = -\mathbb{E} \left[\sum_{i \in \{1..3\}} (w_i LGd(x_i)) \mid \sum_{i \in \{1..3\}} (w_i LGd(x_i)) < q \right] \quad (11)$$

where $LGd(x_i)$ is the loss given distress for the underlying asset value x_i , a function that is calibrated by interpolation between 0 and a loss given default rate of 60 percent.²⁴ The expectation is computed using Monte Carlo integration with 10,000 simulations.

²⁴Typically, losses given default (LGD) are calibrated at 60 percent, but this assumption is not sufficient to calibrate the entire distribution of losses, in particular outside the region of default. Because even outside the region of default valuation losses occur (for instance, because of expectations that the asset is getting closer to the region of default), we set a function for loss given distress: $LGd(x) = LGD$ if $x > X_d^x$; $LGd(x) = 0$ if $x < K_x$; $LGd(x) = \frac{\Phi(K_x) - \Phi(x)}{\Phi(K_x) - \Phi(X_d^x)}$ if $K_x < x < X_d^x$, where Φ is the cumulative distribution function of the returns of x

Following on the work of Tarashev et al. (2009) and Drehmann and Tarashev (2013) on Shapley values in financial systems, we use the measures of expected losses $\mathcal{V}(S)$ to compute the Shapely value of a firm i in a system N :

$$ShV_N(i) = \frac{1}{n} \sum_{n_s=1}^n \frac{1}{C(n_s)} \sum_{\{S \in N \mid card(S)=n_s \text{ and } i \in S\}} (\mathcal{V}(S) - \mathcal{V}(S - \{i\})) \quad (12)$$

This is the weighted average of a firm’s i marginal contribution to losses ($\mathcal{V}(S) - \mathcal{V}(S - \{i\})$) for each subsystem S of N that includes this firm $\{S \subset N \mid card(S) = n_s \text{ and } i \in S\}$ —see Tarashev et al. (2009). Normalizing the Shapley value by asset size provides a measure of *Marginal Contribution to Systemic Risk (MCSR)* of each firm i in the system N :

$$MCSR_i = \frac{ShV_N(i)}{A} \quad (13)$$

7 Application to the banking and non-bank sectors

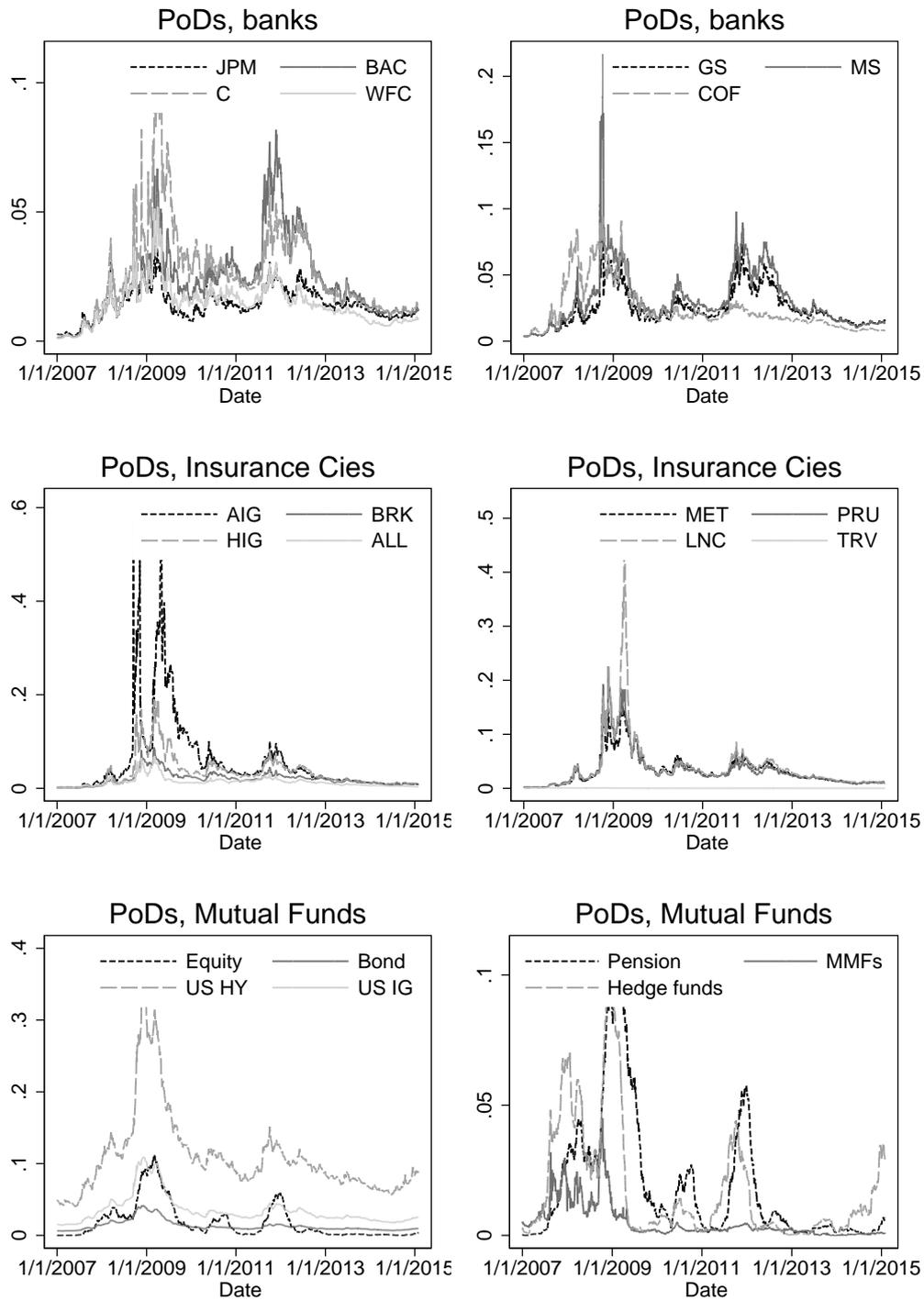
We apply the CIMDO method and compute the different financial stability measures proposed on two datasets of probabilities of distress. The first dataset, centered around the Lehman collapse, is used to assess the extent of contagion at the peak of the crisis, and includes the then-major US universal and investment banks, two insurance companies and the major European banks.²⁵ The second dataset is built to assess the current extent of interconnectedness between the US bank and shadow bank sectors, and includes the major US banks and insurance companies²⁶ as well as indexes for mutual funds (pension funds, money market funds (MMF), US investment grades funds, US High Yields, bond funds, equity funds).

Including the non-bank financial system to this analysis is useful because this sector has been growing for years and contributed to systemic instability, but the lack of data has made its surveillance particularly challenging. Insurance companies can also propagate systemic risk through their nontraditional activities. The insurance sector has

²⁵The list of institutions is Bank of America (BAC), Citi (C), Wachovia (Wacho), Goldman Sachs (GS), Lehman Brothers (LEH), Merrill Lynch (MER), Morgan Stanley (MS), JP Morgan (JPM), AIG and Washington Mutual (WAMU), HSBC, UBS, Deutsche Bank (DB), Barclays (BARC), Credit Suisse (CSFB).

²⁶Wells Fargo (WFC), Citi (C), Bank of America (BAC), JP Morgan (JPM), Morgan Stanley (MS), Goldman Sachs (GS), Capital One Financial (COF), AIG, Allstate (ALL), Prudential Financial (PRU), MetLife (MET), Travelers Companies (TRV), Berkshire Hathaway (BRK), Hartford Financial (HIG)

Figure 3: Probabilities of default of selected institutions and of the different US funds



Source: Bloomberg

increasingly provided bank-like financing, engaging in securities financing transactions, holding corporate bonds, commercial mortgages securities and even providing direct loans to the corporate sector (Acharya and Richardson (2014)). Mutual funds and hedge funds can also transmit shocks because of direct exposure or because of fire sale effects, as shown by Hau and Lai (2017). An example of direct exposure is given by sponsor support. Although sponsors of MMFs (asset managers or banks) do not have necessarily to step in to support their funds, some of them have provided direct support by purchasing the portfolio (e.g. Société Générale bought assets from its MMFs in 2007 and 2008). In addition, MMFs are highly exposed to banks as they are major holders of commercial paper issued by financial firms. For hedge funds, an additional source of contagion for banks comes from prime brokers, who provide margin lending to hedge funds and are also usually subsidiaries of banks.²⁷

7.1 Data

For banks and insurance companies, probabilities of distress were computed using CDS spreads and an assumption of *LGD* of 60 percent.²⁸ For the mutual funds, for which only equity price data was available, the Probabilities of Distress were computed as the probability of the stock price falling below a distress threshold assumed to at the worst 1 percentile of the distribution, where the probability is computed assuming the stock price is log-normal with moments estimated over a 6 months rolling centered window. The data on PoDs, which is the raw data used by CIMDO, is presented in Figure 3.

7.2 Results

Financial institutions are highly interconnected, with distress in one institution associated with high probability of distress elsewhere. The Distress Dependence Matrices for three different days are presented in Tables 2 and 3 and in Figure 4 as a network,²⁹ with the diameter of each vertex proportional to the out-degree of the institution (a measure of average outward spillover, or *systemic importance*, computed as $S_j = \frac{1}{n-1} \sum_{i \neq j} (DiDe)_{i,j}$). The darkness of each node in Figure 4 represents the in-degree (inward spillover, or *vulnerability*, computed as $V_i = \frac{1}{n-1} \sum_{j \neq i} (DiDe)_{i,j}$). For

²⁷See

²⁸There are alternative approaches by which probabilities of distress of individual banks can be empirically estimated. The most well known include the structural approach, PoDs derived from CDS spreads or from out-of-the-money option prices. An extensive empirical analysis of these approaches is presented in Athanasopoulou et al. (2009).

²⁹Pairwise conditional probabilities can be represented as nexus in a network representation

Table 2: Distress Dependence Matrix, major US and European banks (July 1, 2007)

		US banks										European banks					
		Citi	BAC	JPM	Wacho	WAMU	GS	LEH	MER	MS	AIG	BARC	HSBC	UBS	CSFB	DB	Row av.
US banks	Citi	1	0.14	0.11	0.11	0.08	0.09	0.08	0.09	0.09	0.08	0.07	0.07	0.08	0.06	0.07	0.09
	BAC	0.12	1	0.27	0.27	0.11	0.11	0.1	0.12	0.12	0.15	0.08	0.07	0.09	0.06	0.1	0.13
	JPM	0.15	0.42	1	0.31	0.13	0.19	0.16	0.19	0.18	0.17	0.1	0.08	0.12	0.09	0.14	0.17
	Wacho	0.12	0.33	0.24	1	0.11	0.12	0.1	0.12	0.12	0.14	0.07	0.05	0.07	0.05	0.08	0.12
	WAMU	0.16	0.28	0.21	0.23	1	0.12	0.12	0.16	0.13	0.15	0.09	0.08	0.09	0.06	0.09	0.14
	GS	0.17	0.25	0.28	0.21	0.11	1	0.31	0.28	0.31	0.17	0.13	0.11	0.15	0.12	0.18	0.20
	LEH	0.22	0.32	0.32	0.26	0.15	0.43	1	0.35	0.33	0.2	0.14	0.12	0.15	0.14	0.22	0.24
	MER	0.19	0.32	0.33	0.25	0.17	0.33	0.31	1	0.31	0.2	0.15	0.15	0.19	0.15	0.21	0.23
	MS	0.19	0.31	0.28	0.24	0.14	0.35	0.28	0.3	1	0.16	0.14	0.12	0.14	0.12	0.18	0.21
	AIG	0.07	0.14	0.1	0.1	0.05	0.07	0.06	0.07	0.06	1	0.05	0.06	0.07	0.04	0.06	0.07
European banks	BARC	0.04	0.05	0.04	0.04	0.02	0.04	0.03	0.04	0.04	0.04	1	0.18	0.18	0.12	0.12	0.07
	HSBC	0.04	0.04	0.03	0.02	0.02	0.03	0.02	0.03	0.03	0.04	0.16	1	0.13	0.09	0.11	0.06
	UBS	0.04	0.05	0.04	0.03	0.02	0.04	0.03	0.04	0.03	0.04	0.17	0.13	1	0.21	0.15	0.07
	CSFB	0.05	0.06	0.05	0.04	0.03	0.05	0.05	0.06	0.05	0.05	0.19	0.15	0.36	1	0.21	0.10
	DB	0.05	0.09	0.08	0.06	0.03	0.07	0.06	0.07	0.06	0.06	0.17	0.16	0.22	0.19	1	0.10
	Column av.	0.12	0.20	0.17	0.16	0.08	0.15	0.12	0.14	0.13	0.12	0.12	0.11	0.15	0.11	0.14	0.13

Notes: Probability of distress of the bank in the row, conditional on the bank in the column becoming distressed.

Row and column averages exclude diagonal elements

Cells in grey for DiDe > 0.25

each bank, the figure also indicates the eigenvector centrality measure C (a measure of influence in the network), normalized between 0 and 1.

The Distress Dependence Matrices show that links across major financial institutions have increased greatly. On average, if any of the US banks fell into distress, the average probability of another US bank being distressed increased from 27 percent on July 1, 2007 to 41 percent on September 12, 2008. Prior to the financial crisis, no institution seemed vulnerable to other firms' distress, whereas on September 12, 2008, Lehman Brothers had a PoD conditional on any other bank falling into distress averaging of 56 percent. Moreover, the PoD of any other US bank conditional on Lehman falling into distress went from 25 percent on July 1, 2007 to 37 percent on September 12, 2008.³⁰ Links were particularly close between Lehman, AIG, Washington Mutual, and Wachovia, all of which were particularly exposed to housing. JP Morgan, Goldman Sachs, and Bank of America appeared to be the most systemically important institutions, but their vulnerability was relatively low.

Distress in a US bank would have triggered distress in a European bank with an average probability of around only 10 percent. However, links across major European banks

³⁰and reaching respectively 88, 43, and 27 percent for Washington Mutual, AIG, and Wachovia

Table 3: Distress Dependence Matrix, major US and European banks (Sept 12, 2008)

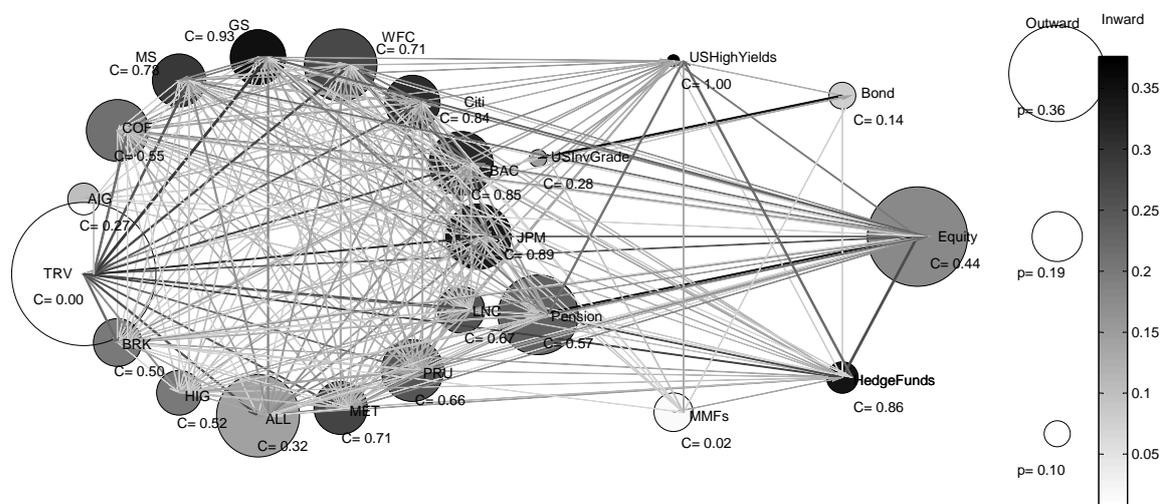
		US banks										European banks					
		Citi	BAC	JPM	Wacho	WAMU	GS	LEH	MER	MS	AIG	BARC	HSBC	UBS	CSFB	DB	Row av.
US banks	Citi	1	0.2	0.19	0.14	0.07	0.17	0.13	0.14	0.16	0.11	0.15	0.17	0.17	0.15	0.16	0.15
	BAC	0.14	1	0.31	0.18	0.05	0.16	0.1	0.13	0.15	0.11	0.12	0.13	0.13	0.11	0.15	0.14
	JPM	0.13	0.29	1	0.16	0.05	0.19	0.11	0.14	0.16	0.09	0.11	0.1	0.12	0.11	0.15	0.14
	Wacho	0.34	0.6	0.55	1	0.17	0.36	0.27	0.31	0.34	0.29	0.27	0.23	0.27	0.25	0.31	0.33
	WAMU	0.93	0.97	0.95	0.94	1	0.91	0.88	0.92	0.91	0.89	0.87	0.86	0.86	0.83	0.86	0.90
	GS	0.15	0.19	0.24	0.13	0.06	1	0.18	0.2	0.27	0.11	0.14	0.13	0.15	0.15	0.19	0.16
	LEH	0.47	0.53	0.58	0.43	0.25	0.75	1	0.59	0.62	0.37	0.39	0.37	0.4	0.42	0.52	0.48
	MER	0.32	0.41	0.47	0.3	0.16	0.53	0.37	1	0.48	0.26	0.31	0.33	0.35	0.35	0.39	0.36
	MS	0.21	0.28	0.29	0.19	0.09	0.4	0.22	0.27	1	0.14	0.18	0.18	0.18	0.18	0.23	0.22
	AIG	0.5	0.66	0.59	0.53	0.29	0.54	0.43	0.49	0.47	1	0.49	0.53	0.53	0.49	0.53	0.51
European banks	BARC	0.1	0.11	0.1	0.08	0.04	0.1	0.07	0.09	0.07	1	0.36	0.31	0.3	0.28	0.15	
	HSBC	0.06	0.06	0.05	0.03	0.02	0.05	0.04	0.05	0.04	0.2	1	0.16	0.16	0.17	0.08	
	UBS	0.11	0.11	0.11	0.07	0.04	0.11	0.07	0.1	0.09	0.08	0.32	0.3	1	0.47	0.34	0.17
	CSFB	0.07	0.07	0.07	0.05	0.03	0.07	0.05	0.07	0.06	0.05	0.2	0.2	0.31	1	0.26	0.11
	DB	0.06	0.08	0.09	0.05	0.03	0.09	0.06	0.07	0.07	0.05	0.18	0.2	0.21	0.24	1	0.11
	Column av.	0.26	0.33	0.33	0.23	0.10	0.32	0.21	0.26	0.28	0.19	0.28	0.29	0.30	0.30	0.32	0.27

Notes: Probability of distress of the bank in the row, conditional on the bank in the column becoming distressed.
 Row and column averages exclude diagonal elements
 Cells in grey for DiDe > 0.25

increased significantly in 2008. UBS appeared to be the European bank under highest stress on that date in our sample, although vulnerabilities in Europe were lower than in the US. In addition, UBS's distress would also have been associated with high stress on Credit Suisse (CSFB) and Barclays (BARC), whose probabilities of distress conditional on UBS becoming distressed were estimated to reach 31 percent. On average, if any of the European banks appeared in distress, the probability of the other European banks being distressed increased from 34 percent on July 1, 2007 to 41 percent on September 12, 2008.

The evolution of the JPoD and the Financial Stability Index (Figure ??, top charts) show how movements in the measures of dependence coincide with events that were considered relevant by the markets on specific dates (events first related to the US financial sector, then to the euro debt crisis). In addition, because distress dependence rises during times of crisis, the measures proposed experience larger increases than those experienced by the PoDs of individual banks, a feature useful to identifying systemic risk. The Financial Stability Index, for instance, shows that during the worst weeks of 2008, the expected number of groups in distress conditional on one being already in distress was 4.5, up from 1.5 before the subprime crisis. The system remains subject to contagion risk, with the FSI being above since 2013.

Figure 4: Distress Dependence Matrix, US banks and non-banks (Mar, 2015)



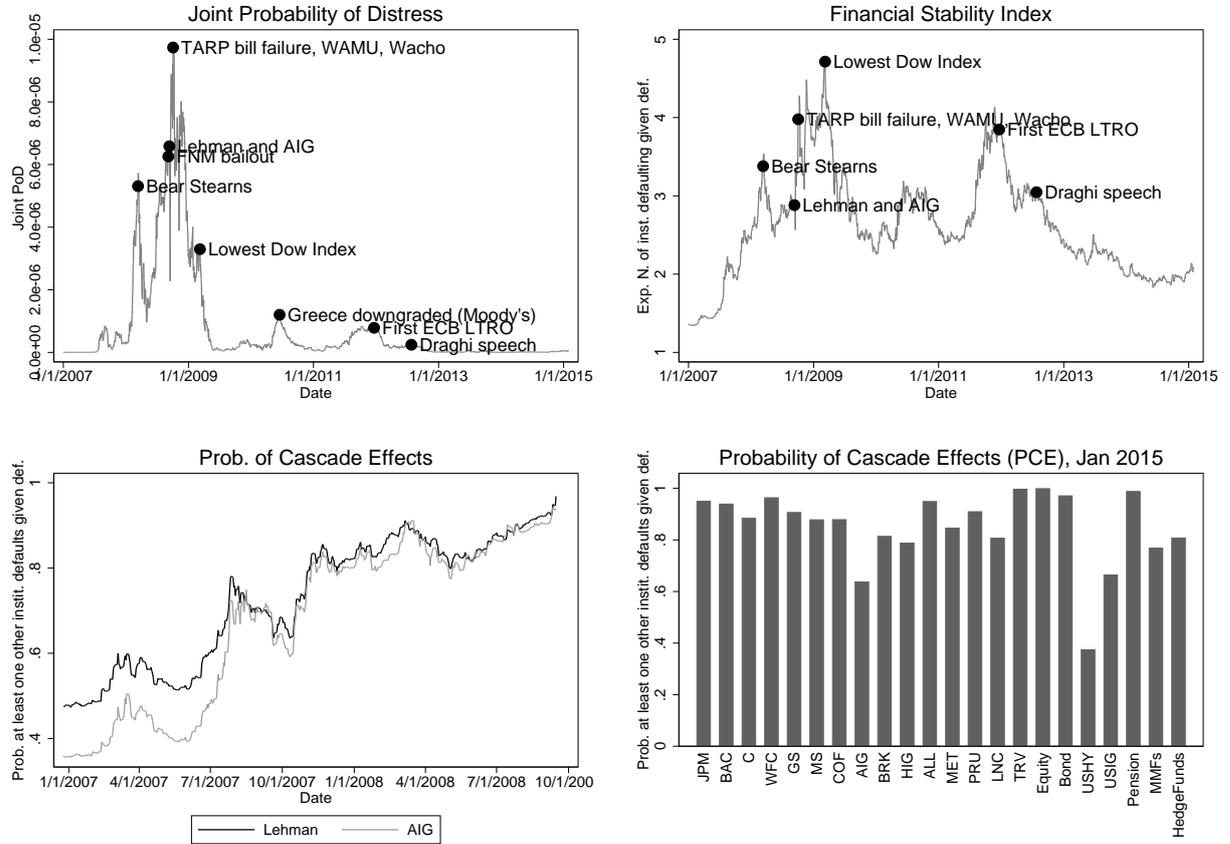
Notes: the size of a disc is proportional to the bank's out-degree centrality; the darkness of the disc is related to the in-degree centrality; C is the eigenvector centrality (a measure of network importance) normalized to [0, 1] ; pairwise conditional probabilities lower than 10 percent were not drawn.

Labels: Wells Fargo (WFC), Citi (C), Bank of America (BAC), JP Morgan (JPM), Morgan Stanley (MS), Goldman Sachs (GS), Capital One Financial (COF), AIG, Allstate (ALL), Prudential Financial (PRU), MetLife (MET), Travelers Companies (TRV), Berkshire Hathaway (BRK), Hartford Financial (HIG), pension funds (Pension), money market funds (MMF), US investment grades funds (USInvGrade), US High Yields (USHighYields), bond funds (Bond), equity funds (Equity)

In the same vein, we estimate that the probability that one or more banks in the system would become distressed, given that Lehman became distressed was 97 percent (Probability of Cascade Effects), up from only 50 percent a year before (Figure ??, bottom left chart). Thus, the domino effect observed in the days after Lehman's collapse was signaled by the Probability of Cascade measure.

The estimates for March 2015 suggest that the probabilities of cascade (PCE) remain high, above 90 percent for several banks (JP Morgan, Bank of America, Wells Fargo, Goldman Sachs), for a few insurance companies (Travelers, Allstate, Prudential) and for the Pension, Equity and Bond funds. This reflects in particular the increased outward spillovers originating from the insurance companies (Tavelers, Allstate) and the Equity and Pension funds, even though funds and insurance companies seem less vulnerable to contagion than other firms, in particular investment banks.

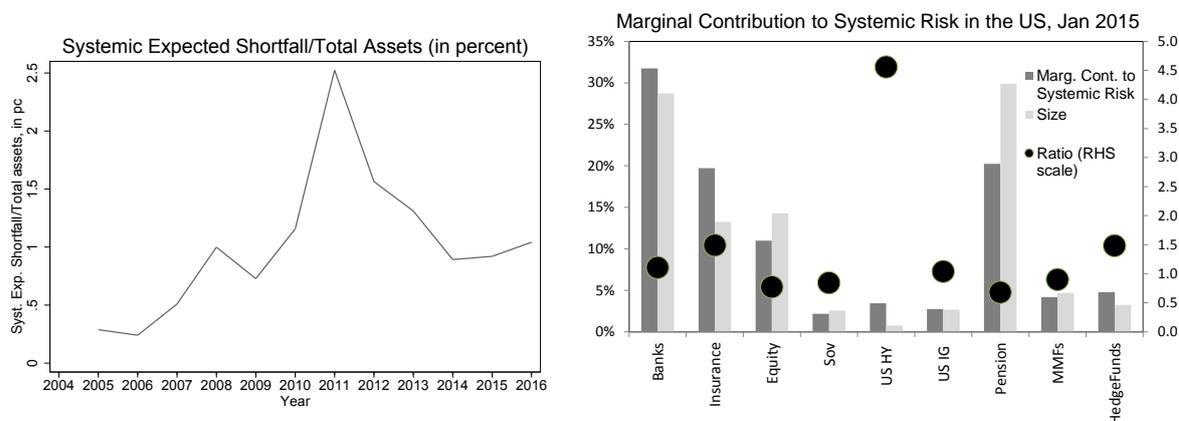
Figure 5: JPoD, FSI and PCE



Source: authors' calculations

Finally, we find that the the Systemic Expected Shortfall would have increase from around 0.5 percent of the financial sector assets before the crisis to around 1 percent of assets in 2016, peaking at 2.5 percent of assets in 2011 (LHS chart of Figure 6). It also appears that in January 2015 the banking sector would have the highest systemic impact in the US, followed by the insurance sector and pension funds (RHS chart of Figure 6). Together, these three sectors' marginal contributions to systemic risk (MCSR) amounted to 73%, with 32% for banks, 25% for insurance sector and 16% for pension funds. To disentangle the role of interconnectedness from that of size, the MCSR is shown along two possible proxies for these factors: asset size, in percent of the financial system total asset size, and the ratio of the MCSR to asset size. The banks and pension funds' contribution to systemic risk can be explained by the size of their balance sheets, whereas interconnectedness is more important for the US high

Figure 6: Expected losses measures of systemic risk



Source: authors' calculations

yields funds, the hedge funds and the insurance companies, three groups for which the contribution to systemic risk was found to be more than proportional to the asset size.

8 Conclusion

This paper considers financial systems as portfolios of entities and presents the CIMDO methodology to infer the multivariate densities that characterize systems' asset values. Data limitations remain an important constraint in the measurement of systemic risk. Given this constraint, CIMDO densities offer important benefits, since they:

- (i) are inferred from the limited data on individual financial entities that is usually readily available;
- (ii) are consistent with the observed probabilities of distress of such entities;
- (iii) outperform parametric distributions frequently employed for risk measurement under the probability integral transformation criterion;
- (iv) can be used to estimate complementary metrics of systemic risk that provide information of alternative perspectives of risk, including measures of tail risk, distress dependence and marginal contribution to systemic risk. Such metrics account for systems interconnectedness structures and incorporate changes in structures when PoDs change. Importantly, the proposed metrics are statistically consistent since they are estimated from a common multivariate density and;

- (v) are easily implementable and can be adapted to cater to a high degree of institutional granularity and data availability. The portfolio assumption allows for the easy incorporation of multiple financial sectors beyond the banking sector into the analysis. Moreover, implementation can be done with market-based data or with publicly available supervisory data. This feature allows an assessment of vulnerabilities developing in sectors where data may be scarce and which are undergoing structural changes. Likewise, estimation can be done in a wide set of countries with heterogeneous data availability.

Improved measurement of systemic risk will remain a priority for financial stability authorities as they work towards integrating the lessons of the financial crisis into their policies, especially macro-prudential policies. In addition to the measures of systemic risk presented in this paper, the proposed multivariate density approach can be useful in further contexts, including the development of macroprudential stress test frameworks and the calibration of theoretical models.

Stress test frameworks have traditionally focused on the assessment of vulnerabilities at the level of individual financial institution. However, in the aftermath of the global financial crisis, efforts have been directed to the development of macroprudential stress tests, which aim to integrate the quantification of losses due to systemic risk amplification mechanisms, especially those coming from indirect interlinkages across entities. The multivariate density characterizing the valuation of financial systems could be used to estimate such losses.

Regarding the calibration of theoretical models, statistical moments obtained from the multivariate distribution of asset values could be used to calibrate theoretical models in a realistic and feasible manner (akin to what has been done in macroeconomics, where reduced-form empirical moments are used to calibrate DSGE models); hence, allowing to benefit from insights brought by the theoretical models with realistic calibrations provided by the empirical models.

Overall, the main advantage of multivariate density approaches is that they allow to focus on different statistical moments and importantly, at the tail of the densities that characterize tail risks, which are essential for the analysis of financial stability. We believe that this field would benefit from the multivariate approach going forward.

Appendix

Proof of Proposition 2.

Assume $(\mu, \lambda_x, \lambda_y)$ solve system (14):

$$\begin{cases} \int \int q(x, y) \exp(-1 - \mu - \lambda_x \chi_{[X_d^x, +\infty)} - \lambda_y \chi_{[X_d^y, +\infty)}) \chi_{[X_d^x, \infty)} dx dy = PoD_t^x \\ \int \int q(x, y) \exp(-1 - \mu - \lambda_x \chi_{[X_d^x, +\infty)} - \lambda_y \chi_{[X_d^y, +\infty)}) \chi_{[X_d^y, \infty)} dx dy = PoD_t^y \\ \int \int q(x, y) \exp(-1 - \mu - \lambda_x \chi_{[X_d^x, +\infty)} - \lambda_y \chi_{[X_d^y, +\infty)}) dx dy = 1 \end{cases} \quad (14)$$

In addition $(\hat{\mu}_1, \hat{\lambda}_x)$ solve the system (15) whilst $(\hat{\mu}_2, \hat{\lambda}_y)$ solve the system (16)

$$\begin{cases} \int q(x) \exp(-1 - \hat{\mu}_1 - \hat{\lambda}_x \chi_{[X_d^x, +\infty)}) \chi_{[X_d^x, \infty)} dx = PoD_t^x \\ \int q(x) \exp(-1 - \hat{\mu}_1 - \hat{\lambda}_x \chi_{[X_d^x, +\infty)}) dx = 1 \end{cases} \quad (15)$$

$$\begin{cases} \int q(y) \exp(-1 - \hat{\mu}_2 - \hat{\lambda}_y \chi_{[X_d^y, +\infty)}) \chi_{[X_d^y, \infty)} dy = PoD_t^y \\ \int q(y) \exp(-1 - \hat{\mu}_2 - \hat{\lambda}_y \chi_{[X_d^y, +\infty)}) dy = 1 \end{cases} \quad (16)$$

We define the different probabilities of default/non-default under the prior distribution q as

$$\begin{aligned} Q_{xy} &= \int_{X_d^x}^{+\infty} \int_{X_d^y}^{+\infty} q(x, y) dy dx, & Q_{x\bar{y}} &= \int_{X_d^x}^{+\infty} \int_{-\infty}^{X_d^y} q(x, y) dy dx, \\ Q_{\bar{x}y} &= \int_{-\infty}^{X_d^x} \int_{X_d^y}^{+\infty} q(x, y) dy dx, & Q_{\bar{x}\bar{y}} &= \int_{-\infty}^{X_d^x} \int_{-\infty}^{X_d^y} q(x, y) dy dx, \\ Q_x &= \int_{X_d^x}^{+\infty} q(x) dx, & Q_{\bar{x}} &= 1 - Q_x = \int_{-\infty}^{X_d^x} q(x) dx, \\ Q_y &= \int_{X_d^y}^{+\infty} q(y) dy, & Q_{\bar{y}} &= 1 - Q_y = \int_{-\infty}^{X_d^y} q(y) dy \end{aligned}$$

We separate the indefinite integrals according to the interval of indicator functions:

$$\begin{cases} e^{-1-\mu} (e^{-\lambda_x} e^{-\lambda_y} Q_{xy} + Q_{x\bar{y}} e^{-\lambda_x}) = PoD^x \\ e^{-1-\mu} (e^{-\lambda_x} e^{-\lambda_y} Q_{xy} + Q_{\bar{x}y} e^{-\lambda_y}) = PoD^y \\ e^{-1-\mu} (e^{-\lambda_x} e^{-\lambda_y} Q_{xy} + Q_{\bar{x}\bar{y}} e^{-\lambda_x} + Q_{x\bar{y}} e^{-\lambda_x} + Q_{\bar{x}y} e^{-\lambda_y}) = 1 \end{cases} \quad (17)$$

which can be rewritten

$$\begin{cases} \lambda_x = -1 - \mu - \ln(PoD^x) + \ln(Q_{xy} e^{-\lambda_y} + Q_{x\bar{y}}) \\ \lambda_y = -1 - \mu - \ln(PoD^y) + \ln(Q_{xy} e^{-\lambda_x} + Q_{\bar{x}y}) \\ \mu = -1 + \ln(Q_{xy} e^{-\lambda_x} e^{-\lambda_y} + Q_{\bar{x}\bar{y}} e^{-\lambda_x} + Q_{x\bar{y}} e^{-\lambda_x} + Q_{\bar{x}y} e^{-\lambda_y}) \end{cases} \quad (18)$$

Similary, the univariate solution implies

$$\begin{cases} \hat{\lambda}_x &= -1 - \hat{\mu}_1 - \ln(PoD^x) + \ln(Q_x) \\ \hat{\mu}_1 &= -1 + \ln(Q_x e^{-\hat{\lambda}_x} + Q_{\bar{x}}) \end{cases} \quad (19)$$

Thus

$$\begin{aligned} \lambda_x - \hat{\lambda}_x &= \ln(Q_x e^{-\hat{\lambda}_x} + Q_{\bar{x}}) - \ln(Q_{xy} e^{-\lambda_x} e^{-\lambda_y} + Q_{\bar{x}y} e^{-\lambda_y} + Q_{x\bar{y}} e^{-\lambda_x} + Q_{\bar{x}\bar{y}}) \\ &\quad + \ln(Q_{xy} e^{-\lambda_y} + Q_{x\bar{y}}) - \ln(Q_x) \end{aligned}$$

Let us approximate w.r.t. to λ_y and λ_x , assuming the lagrange multipliers are small, i.e. write $e^{-\lambda_y} \approx 1 - \lambda_y$ and $e^{-\lambda_x} \approx 1 - \lambda_x$:

$$\begin{aligned} \lambda_x - \hat{\lambda}_x &\approx \ln(Q_x(1 - \hat{\lambda}_x) + Q_{\bar{x}}) - \ln[Q_{xy}(1 - \lambda_x - \lambda_y) + Q_{\bar{x}y}(1 - \lambda_y) \\ &\quad + Q_{x\bar{y}}(1 - \lambda_x) + Q_{\bar{x}\bar{y}}] + \ln(Q_{xy}(1 - \lambda_y) + Q_{x\bar{y}}) - \ln(Q_x) \end{aligned}$$

Since $Q_{xy} + Q_{x\bar{y}} = Q_x$, $Q_{\bar{x}y} + Q_{\bar{x}\bar{y}} = Q_{\bar{x}}$, $Q_x + Q_{\bar{x}} = 1$ and $Q_{xy} + Q_{\bar{x}y} = Q_y$,

$$\begin{aligned} \lambda_x - \hat{\lambda}_x &\approx \ln(1 - Q_x \hat{\lambda}_x) - \ln(1 - \lambda_x Q_x - \lambda_y Q_y) \\ &\quad + \ln(Q_x - \lambda_y Q_{xy}) - \ln(Q_x) \end{aligned}$$

Note also that $Q_{xy} \ll Q_x$, $Q_y \ll 1$ implies $\lambda_x - \hat{\lambda}_x \approx -Q_x \lambda_x + \lambda_x Q_x + \lambda_y Q_y - \lambda_y Q_{xy} / Q_x$

Thus, $\lambda_x - \hat{\lambda}_x = \lambda_y \frac{Q_x Q_y - Q_{xy}}{Q_x Q_{\bar{x}}}$ and by symmetry $\lambda_y - \hat{\lambda}_y = \lambda_x \frac{Q_x Q_y - Q_{xy}}{Q_y Q_{\bar{y}}}$.

This proves that

$$\lambda_x \approx \frac{\hat{\lambda}_x + \hat{\lambda}_y \frac{Q_x Q_y - Q_{xy}}{Q_x Q_{\bar{x}}}}{1 - \frac{(Q_x Q_y - Q_{xy})^2}{Q_x Q_{\bar{x}} Q_y Q_{\bar{y}}}} \quad (20)$$

□

Proof of Proposition 3.

For a bivariate t distribution, the probability density function is

$$q(t_1, t_2) = \frac{|D|^{1/2}}{2\pi} (1 + (D_{11}t_1^2 + 2D_{12}t_1t_2 + D_{22}t_2^2)/\nu)^{-(\nu+2)/2},$$

where

$$D = \Sigma^{-1} = \begin{pmatrix} 1 & \sigma \\ \sigma & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{1-\sigma^2} & -\frac{\sigma}{1-\sigma^2} \\ -\frac{\sigma}{1-\sigma^2} & \frac{1}{1-\sigma^2} \end{pmatrix}.$$

Then we have

$$q(t_1, t_2) = \frac{1}{2\pi\sqrt{1-\sigma^2}} \left(1 + \frac{t_1^2 - 2\sigma t_1 t_2 + t_2^2}{\nu(1-\sigma^2)}\right)^{-(\nu+2)/2},$$

The Taylor expansions of $f(t_1, t_2)$ with respect to σ around 0 is

$$q(t_1, t_2) = \frac{1}{2\pi} \left(1 + \frac{t_1^2 + t_2^2}{\nu}\right)^{-\frac{\nu+2}{2}} + \frac{(\nu+2)\nu^{1+\frac{\nu}{2}}}{2\pi} t_1 t_2 (\nu + t_1^2 + t_2^2)^{-\frac{\nu}{2}-2} \sigma + O(\sigma^2) \quad (21)$$

The first term on the right hand side of the above equation is the value of $q(t_1, t_2)$ with $\sigma = 0$. Substitute equation (21) into equations (18), and let \tilde{Q}_{ij} be the probabilities of default/non-default for a prior bivariate t distribution with identity correlation matrix. Then

$$\begin{aligned} Q_{xy} &= \tilde{Q}_{xy} + \int_{X_d^x}^{+\infty} \int_{X_d^y}^{+\infty} \frac{(\nu+2)\nu^{1+\frac{\nu}{2}}}{2\pi} t_1 t_2 (\nu + t_1^2 + t_2^2)^{-\frac{\nu}{2}-2} dt_1 dt_2 \sigma + O(\sigma^2) \\ &= \tilde{Q}_{xy} + J \sigma + O(\sigma^2), \end{aligned}$$

$$\begin{aligned} Q_{x\bar{y}} &= \tilde{Q}_{x\bar{y}} + \int_{X_d^x}^{+\infty} \int_{-\infty}^{X_d^y} \frac{(\nu+2)\nu^{1+\frac{\nu}{2}}}{2\pi} t_1 t_2 (\nu + t_1^2 + t_2^2)^{-\frac{\nu}{2}-2} dt_1 dt_2 \sigma + O(\sigma^2) \\ &= \tilde{Q}_{x\bar{y}} - J \sigma + O(\sigma^2), \end{aligned}$$

$$\begin{aligned} Q_{\bar{x}y} &= \tilde{Q}_{\bar{x}y} + \int_{-\infty}^{X_d^x} \int_{-\infty}^{X_d^y} \frac{(\nu+2)\nu^{1+\frac{\nu}{2}}}{2\pi} t_1 t_2 (\nu + t_1^2 + t_2^2)^{-\frac{\nu}{2}-2} dt_1 dt_2 \sigma + O(\sigma^2) \\ &= \tilde{Q}_{\bar{x}y} - J \sigma + O(\sigma^2), \end{aligned}$$

$$\begin{aligned} Q_{\bar{x}\bar{y}} &= \tilde{Q}_{\bar{x}\bar{y}} + \int_{-\infty}^{X_d^x} \int_{X_d^y}^{+\infty} \frac{(\nu+2)\nu^{1+\frac{\nu}{2}}}{2\pi} t_1 t_2 (\nu + t_1^2 + t_2^2)^{-\frac{\nu}{2}-2} dt_1 dt_2 \sigma + O(\sigma^2) \\ &= \tilde{Q}_{\bar{x}\bar{y}} + J \sigma + O(\sigma^2), \end{aligned}$$

where $J = \frac{\nu^{\nu/2}}{2\pi} (\nu + X_d^{x2} + X_d^{y2})^{-\nu/2}$. Now we can represent the solution (18) of CIMDO, in terms of \tilde{Q}_{ij} and correlation σ , as the following

$$\begin{aligned} \lambda_x &= -1 - \mu - \ln(PoD_x) + \ln(\tilde{Q}_{xy} e^{-\lambda_y} + \tilde{Q}_{x\bar{y}} + (e^{-\lambda_y} - 1)J\sigma + O(\sigma^2)) \\ \lambda_y &= -1 - \mu - \ln(PoD_y) + \ln(\tilde{Q}_{xy} e^{-\lambda_x} + \tilde{Q}_{\bar{x}y} + (e^{-\lambda_x} - 1)J\sigma + O(\sigma^2)) \quad (22) \\ \mu &= -1 + \ln[\tilde{Q}_{xy} e^{-\lambda_x} e^{-\lambda_y} + \tilde{Q}_{\bar{x}y} e^{-\lambda_y} + \tilde{Q}_{x\bar{y}} e^{-\lambda_x} + \tilde{Q}_{\bar{x}\bar{y}} + \\ &\quad (e^{-\lambda_x} e^{-\lambda_y} - e^{-\lambda_y} - e^{-\lambda_x} + 1)J\sigma + O(\sigma^2)] \end{aligned}$$

In equations (22), the terms in σ and in $O(\sigma^2)$ of represent the effect of the non-identity correlation matrix on the solution of CIMDO. Note that $Q \rightarrow 0$ as $X_d^x, X_d^y \rightarrow +\infty$ and $e^{-\lambda_x}, e^{-\lambda_y} \rightarrow 1$ as $\lambda_x, \lambda_y \rightarrow 0$. Both results indicate that a possible mis-specification

of the prior correlation is less important the lower the default probabilities and the closest the prior is from being consistent with the observed PoDs. \square

Proof of Proposition 4.

Define the Probability Integral Transform under the distribution f as $P(x) = \int_{-\infty}^x f(t)dt$. Then, define u and v as

$$\begin{aligned} u &= P(x) \iff x = P^{(-1)}(u) \\ v &= P(y|x) \iff y = P^{(-1)}(v|x) \end{aligned}$$

Lemma 1: u, v are independent

Proof: In order to prove the independence assumption, we know that the joint density $c[u, v]$ is defined under the distribution of transformations of random variables as

$$c[u, v] = f [P^{(-1)}(u), P^{(-1)}(v|x)] \cdot \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right|$$

Since in this case

$$\begin{aligned} u &= P(x) \iff x = P^{(-1)}(u) \implies \frac{\partial x}{\partial u} = f [P^{(-1)}(u)]^{-1} \\ v &= P(y|x) \iff y = P^{(-1)}(v|x) \implies \frac{\partial y}{\partial v} = f [P^{(-1)}(v|x)/x]^{-1} \\ \frac{\partial x}{\partial v} &= \frac{\partial y}{\partial u} = 0 \end{aligned}$$

therefore we get

$$\begin{aligned} c[u, v] &= f [P^{(-1)}(u), P^{(-1)}(v|x)] \cdot \frac{1}{f [x] \cdot f [y|x]} \\ c[u, v] &= f [x, y] \cdot \frac{1}{f [x] \cdot f [y|x]} \\ c[u, v] &= f(x) \cdot f(y|x) \cdot \frac{1}{f [x] \cdot f [y|x]} \\ c[u, v] &= 1 \end{aligned} \tag{23}$$

which proves that u, v are independent. \square

Lemma 2: If f is the true distribution (i.e. if $f = p$), then u, v are distributed $\mathbf{U}(0, 1)$.

Proof: Let $F(x) = \int_{-\infty}^x f(t)dt$. For u on $[0, 1]$, we have:

$$\begin{aligned} \mathbb{P}[U \leq u] &= \mathbb{P}[F(x) \leq u] \\ &= \mathbb{P}[F^{-1}[F(x)] \leq F^{-1}(u)] \\ &= \mathbb{P}[X \leq F^{-1}(u)] = F[F^{-1}(u)] = u \end{aligned}$$

For $u < 0$, $\mathbb{P}[U < u] = 0$ and for $u > 1$, $\mathbb{P}[U > u] = 0$ since the range of a *cdf* is $[0, 1]$. Thus $U \sim \mathbf{U}(0, 1)$.

Calibration of competing distributions

Normal distribution: The volatility parameters obtained are $\sigma_x = 1.3422$; $\sigma_y = 1.5864$.

t-distribution: The volatility parameters obtained are $\sigma_x = 1.5353$; $\sigma_y = 1.8386$.

Mixture distribution: $Pdf_{Mixture} = \int_{X_d^y}^{\infty} \int_{X_d^x}^{\infty} \{ \mathbf{Pr}_{o1} [N_1(\mu_1, \Sigma_1)] + \mathbf{Pr}_{o2} [N_2(\mu_2, \Sigma_2)] \} dx dy =$

PoD = [0.22, 0.29]. $\mathbf{Pr}_{o1} = 0.7817$, $\mathbf{Pr}_{o2} = 0.2183$ are the values indicating the probabilities of the *quiet* and *volatile* states, N_1 , N_2 are bivariate normal distributions under the *quiet* and *volatile* states, $\mu_1 = [0, 0]$, $\mu_2 = [0.3, 0.3]$ are the mean borrowers' asset values under the *quiet* and *volatile* states and $\Sigma_1 = \begin{bmatrix} 1.0000 & 0 \\ 0 & 1.5104 \end{bmatrix}$,

$\Sigma_2 = \begin{bmatrix} 100.0000 & 0 \\ 0 & 109.1398 \end{bmatrix}$ are variance covariance matrices for the bivariate distribution under the *quiet* and *volatile* states.

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