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OPTIMAL CAPITAL GROWTH WITH CONVEX SHORTFALL PENALTIES

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Abstract

The optimal capital growth strategy or Kelly strategy, has many desirable properties such as maximizing the asympotic long run growth of capital. However, it has considerable short run risk since the utility is logarithmic, with essentially zero Arrow-Pratt risk aversion. Most investors favor a smooth wealth path with high growth. In this paper we provide a method to obtain the maximum growth while staying above a predetermined ex-ante discrete time smooth wealth path with high probability, with shortfalls below the path penalized with a convex function of the shortfall so as to force the investor to remain above the wealth path. This results in a lower investment fraction than the Kelly strategy with less risk, and lower but maximal growth rate under the assumptions.

A mixture model with Markov transitions between several normally distributed market regimes is used for the dynamics of asset prices. The investment model allows the determination of the optimal constrained growth wagers at discrete points in time in an attempt to stay above the ex-ante path.

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1 Introduction

In this paper we provide a method to deal with the risky short run properties of the expected log capital growth criterion . That theory provides the maximum long run asympotic growth, as shown in increasing generality by Kelly (1956), Breiman (1961), Algeot and Cover (1988) and Thorp (2006); see also the book MacLean, Thorp and Ziemba (2010), which discusses the Kelly strategy and reprints the major papers on the topic. However, the Arrow-Pratt risk aversion index for the logarithmitic utility function is $\frac{1}{w}$, which is essentially zero. Consequently, the wealth trajectories for the Kelly investor are very volatile and risky. One way to deal with the risk is to use fractional Kelly strategies which blend the Kelly portfolio with cash; see MacLean, Blazenko and Ziemba (1992). But this is ad hoc and does not generally produce a smooth wealth path. It reduces risk and growth, so the wealth trajectory has the same dynamics but with a smaller scale. To get a smooth trajectory the fraction in the Kelly may need to be very low and correspondingly the growth rate is small. Applications that use the fractional approach are Grauer and Hakansson (1986, 1987), Hausch, Ziemba and Rubinstein (1981), and Mulvey and Valdmirou (1992); see also the survey of applications in Ziemba (2013).

Our approach is to specify a desired ex-ante wealth path at discrete decision points in time, and to maximize the growth rate (objective) while staying above the path (constraint). This cannot be achieved with certainty, so a condition to exceed the desired path with high probability is imposed and path violations/shortfalls are penalized in the objective. The penalty is a convex function of the shortfall, so that, for example, doubling the shortfall incurs a more than doubled negative penalty. The decision model has strong theoretical underpinnings linked to Prospect Theory (Tversky and Kahneman, 1993) and is analytically attractive as a stochastic dynamic optimization program.

In Section 2 the financial market is presented. The setting is a Markov regime switching framework, with geometric Brownian motion prices within regime. The resulting log prices are a mixture of normals and provide the flexibility needed to obtain accurate price predictions as inputs to investment decisions. The investment model is developed in Section 3, where the objective is to maximize growth penalized for shortfalls, subject to a dynamic VaR constraint requiring wealth to exceed a specified benchmark path with high probability. The unconstrained Kelly strategy is analyzed in Section 4, where the shortfall rate and shortfall size relative to a benchmark are developed. The results provide context for the penalized shortfall approach which is dealt with in Section 5. Section 6 concludes with observations and implications.

2 Market Structure

The wealth accumulation process is a stochastic dynamic system which depends on the allocation of capital to investment opportunities and the changing prices of those assets. A standard model for price dynamics is geometric Brownian motion or a geometric random walk in discrete time. It is known that this model fails to capture important characteristics of asset prices, notably price distributions which are not log-normal and time dependent volatility. A flexible framework which accomodates observed price behavior is a Markov regime switching model, where the dynamics within a regime follow the standard geometric random walk and the parameters in the dynamics vary by regime. Hamilton (1989) successfully applied the Markov model to US GDP data and charcterized the changing pattern of the US economy. Ang and Bekaert (2002) used regime shifts in a study of international asset allocation. Guidolin and Timmermann (2006) provided important insights into how investments vary across market regimes. The regimes make economic sense, and the regime switching market structure is very amenable to analysis.

2.1 Switching Regime Model

Consider a competitive financial market with n assets whose prices are stochastic dynamic processes, and a single asset whose price is non-stochastic. Let the vector of prices at time

t be

$$P(t) = (P_0(t), P_1(t), ..., P_n(t))', \tag{1}$$

where $P_0(t)$ is the price of the risk free asset, with rate of return r_t at time t. It is assumed that the financial market is separated into m distinct regimes. Suppose the market is in regime k at time t, and let $Y_{ik}(t) = \ell n P_{ik}(t), i = 0, ..., n$ be the log-prices in regime $k, k = 1, ..., m$.

The price dynamics within regime k are defined by the stochastic differential equations

$$dY_{0k}(t) = r_t dt \tag{2}$$

$$dY_k(t) = \alpha_k dt + \Delta_k dZ_k, k = 1, ..., m, \tag{3}$$

with $Y_k(t) = \begin{pmatrix} Y_{1k}(t) \\ \vdots \\ Y_{nk}(t) \end{pmatrix}, \alpha_k = \begin{pmatrix} \alpha_{1k} \\ \vdots \\ \alpha_{nk} \end{pmatrix}, \Delta_k = (\delta_{ijk}), dZ_k = \begin{pmatrix} dZ_{1k} \\ \vdots \\ dZ_{nk} \end{pmatrix}$, where $dZ_{ik}, i = 1, ..., n$ are independent Brownian motions.

In this framework the risky asset prices within a regime are assumed to have a joint log-normal distribution.

The regimes over time $\{S(t), t > 0\}$ follow a discrete state continuous time Markov process. The state space is finite $S = \{S_1, ..., S_m\}$ and states will be referred to as regimes: $\{i = 1, ..., m\}$. The dynamics of the Markov process are driven by the intensity g_{ij}, which is the rate of transitioning from regime i to regime j. The rate of switching from regime i at time t to regime j at time $t + h$ is $P[S(t + h) = j|S(t) = i] = g_{ij} \cdot h + o(h)$, where $\frac{o(h)}{h} \to 0$ as $h \to 0$. If the process is in regime i it transitions out of i to another regime with rate $g_i = \sum_{j=1}^{m} g_{ij}$. Then $p_{ij} = \frac{g_{ij}}{g_i}$ is the probability that the process moves to regime j from regime i. For regimes i, j the transition probability function $P_{ij}(t) = Pr[S(t) = j|X(0) = i]$ is a continuous function of t. This function satisfies the Chapman-Kolmogorov equations

$P_{ij}(t+s) = \sum_{k \in S} P_{ik}(t) P_{kj}(s)$.

The market structure given by the switching factor model has advantages: (i) the ability to estimate the parameters in the model from observations on the asset returns and (ii) the ability to define analytically tractable investment models. The standard estimation procedure is the Expectation Maximization (EM) algorithm (Dempster et al, 1977). The investment model developed in subsequent sections considers that parameter values are known/estimated and focuses on investment strategies which control risk. The aspect of risk which is attributable to estimation error is not considered, but the positive results in the literature with the Markov switching model and the EM algorithm are the basis for the defined market structure.

2.2 Wealth Equations

The prices on assets vary continuously. The decision model assumes that time is divided into equal size planning intervals. A decision on the fractions of investment capital allocated to assets is made at the beginning of a period and those fractions are fixed for the period, although continuous rebalancing of allocated capital is required to maintain the fixed fractions. At the begining of the next period the decision fractions are updated. In the analysis of trading strategies, the following assumtions are made:

1. All assets have limited liability.
2. There are no transactions costs, taxes, or problems with indivisibility of assets.
3. Capital can be borrowed or lent at the risk free interest rate at any level.
4. Short sales of all assets is allowed.

Consider that the decision points are $t_1 = 0, t_2 = t_1 + d, ..., t_{L+1} = t_L + d = T$. An investment strategy is the vector process

$$\{(x_0(t_l), X(t_l)), l = 1, ..., L\} = \{(x_0(t_l), x_1(t_l), ..., x_n(t_l)), l = 1, ..., L\} \tag{4}$$

where $\sum_{i=0}^{n} x_i(t_l) = 1$ for any t_l, with $x_0(t_l)$ the investment fraction in the risk-free asset and $x_i(t_l)$ the fraction invested in risky asset $i, i = 1, ..., n$.

The change in wealth from an investment decision $X(t)$ is determined by the changes in prices, which depend on the Markov regime switching process and the dynamics of prices within a regime. Let $\Sigma_k^2 = \Sigma_k' \Sigma_k$ and $\phi_k = \alpha_k + \frac{1}{2}\Sigma_k^2 e, k = 1, ..., m$. Then the instantaneous change in wealth if the market is in regime k is

$$dW_k(t) = [X'(t)(\phi_k - re) + r]W_k(t)\,dt + W_k(t)[X'(t)\Sigma_k dZ_{jk}]. \tag{5}$$

If the wealth at time t is w_t and the investment decision is maintained through rebalancing as a fixed fraction from time t to time $t+h$, then the accumulated wealth is

$$W_k(t+h) = w_t \cdot exp\left\{[X'(t)(\phi_k - re) + r - \frac{1}{2}X'(t)\Sigma_k^2 X(t)]h + h^{\frac{1}{2}} X(t)'\Sigma_k Z_k\right\}, \tag{6}$$

where $Z_k' = (Z_{1k}, ..., Z_{nk}), Z_{ik} \sim N(0,1)$.

The following assumptions are made for wealth dynamics between decision points:

1. *There is at most one regime transition in the time interval* $(t, t+d)$. Given regime i at time t, then $\tau_i =$ the time to switch from regime i to another regime is Exponential with parameter q_i, and $Pr[\tau_i \leq d] = e^{-q_i d} \approx q_i d$, which is small for a short time interval. For two transitions $Pr[\tau_i + \tau_j \leq d] \approx q_i q_j d^2$, a negligible quantity.

2. *If there is a transition it occurs at the start of the interval* $(t, t+d)$. The probability that there is one transition in the interval and it is from i to j is $P_{ij}(d) = Pr[S(t+d) = j | S(t) = i] = Pr[S(d) = j | S(0) = i] \approx q_i d \times p_{ij}$. Then the chance of remaining in regime i is $P_{ii}(d) \approx 1 - q_i d$. Suppose the transition from i to j occurred at time $h, h \leq d$. Then accumulated wealth on the interval with the fixed fraction strategy $X(t)$ is $w_t \cdot exp\left\{[X'(t)(\phi_i - re) + r - \frac{1}{2}X'(t)\Sigma_i^2 X(t)]h + h^{\frac{1}{2}} X(t)'\Sigma_i Z_i\right\} \times$

$exp\left\{[X'(t)(\phi_j - re) + r - \frac{1}{2}X'(t)\Sigma_j^2 X(t)](d-h) + (d-h)^{\frac{1}{2}}X(t)'\Sigma_j Z_j\right\}$. The expected rate of growth is $ln(w_t) + E\left\{[X'(t)(\phi_j - re) + r - \frac{1}{2}X'(t)\Sigma_j^2 X(t)]d\right\} + D(h)$, where $D(h) = E\left\{[X'(t)(\phi_i - \phi_j) - \frac{1}{2}X'(t)[\Sigma_i^2 - \Sigma_j^2]X(t)]h\right\}$. Although there is a slight chance of switching to markedly different regimes, the more likely scenario is a switch to an adjacent/close regime. Also τ_i, the time in regime i, is Exponential with density $\kappa_i(h) = q_i e^{-q_i h}$ and smaller values are more likely. So the value of $D(h)$ is small and the rate of return in the next time interval assuming there is a regime switch to j is close to $ln(w_t) + E\left\{[X'(t)(\phi_j - re) + r - \frac{1}{2}X'(t)\Sigma_j^2 X(t)]d\right\}$, the rate from regime j over the interval.

3. The investment strategy is a fixed fraction of wealth throughout the interval, with continuous rebalancing to maintain the fractions.

The implication of these assumptions is that a decision on investment fractions for the next interval are based on the current position and the probability that one of the regimes: $i, i = 1, ..., m$ will prevail for the entire interval.

Consider that the wealth at the beginning of period t is w_{t-1}, the regime is k in period t, and the period length is subsumed into the parameters. That is, if the period is one day, the parameter $\tilde{\phi}_k = \phi d$ is the vector of expected daily returns , $\tilde{\Sigma}_k = \Sigma d^{\frac{1}{2}}$ is the covariance matrix for daily returns and $\tilde{r} = rd$ is the one day risk free return. Then the conditional wealth at the end of period t if the regime is k, given the fixed investment strategy $X(t)$, is

$$W_k(t) = w_{t-1} \cdot exp\left\{[X'(t)(\tilde{\phi}_k - \tilde{r}e) + \tilde{r} - \frac{1}{2}X'(t)\tilde{\Sigma}_k^2 X(t)] + X(t)'\tilde{\Sigma}_k Z_k\right\}. \quad (7)$$

Let

$$R_k(X(t)) = exp\left\{[X'(t)(\tilde{\phi}_k - \tilde{r}e) + \tilde{r} - \frac{1}{2}X'(t)\tilde{\Sigma}_k^2 X(t)] + X(t)'\tilde{\Sigma}_k Z_k\right\} \quad (8)$$

be the return on investment $X(t)$ in assets in period t. The rate of return in regime k is

$$ln(R_k(X(t))) = \left[X'(t)(\tilde{\phi}_k - \tilde{r}e) + \tilde{r} - \frac{1}{2}X'(t)\tilde{\Sigma}_k^2 X(t)\right] + X'(t)\tilde{\Sigma}_k Z_k. \quad (9)$$

Let $f_k(v|t), k = 1, ..., m$ be the normal density of $ln(R_k(X(t))$, the log-return given the regime is k. Then $E(ln(R_k(X(t))) = \mu_k(t)$ and $\sigma(ln(R_k(X(t))) = \sigma_k(t)$, where $\mu_k(t) = X'(t)(\phi_k - re) + r - \frac{1}{2}\sigma_k^2(t)$ and $\sigma_k^2(t) = X'(t)\Delta_k^2 X(t)$. For the transition probability function $P_{ij}(d)$ the interval time d is fixed, so we will drop the time d and simply refer to the fixed matrix $P = (P_{ij})$. Assume that the distribution over regimes is $\pi(t) = (\pi_1(t), ..., \pi_m(t))$, where $\pi(t) = \pi(t-1)P$. Let the unconditional rate of return on investment $X(t)$ be $ln\,(R(X(t)))$. Then the unconditional distribution for $ln\,(R(X(t)))$ is a mixture of normals $f(v|t) = \pi_1(t)f_1(v|t) + \ldots + \pi_m(t)f_m(v|t)$.

Based on investment decisions at discrete points in time $t = 1, ..., T$, the wealth process and the rate of return process are analyzed as discrete time stochastic processes. For the stochastic process $R(X(t))$ a trajectory of the data process is associated with an outcome ω in the space Ω of all returns trajectories at times $t = 1, ..., T$. The distributions over returns at each time t generate a probability measure P on Ω and the associated probability space (Ω, B, P). The sample space can be represented as $\Omega = \Omega_1 \times \cdots \times \Omega_T$, with $\omega_t \in \Omega_t$ the data at time t and Ω^t the data up to and including time t. Subsets of Ω are of the form $A = A_1 \times \ldots \times A_T$. Let $R(\omega, X(t)), \omega \in \Omega$, be a returns trajectory, where $X(t)$ is an investment strategy which can depend on the data history but not on unknown future returns. The discrete time wealth trajectory is $W(\omega, t) = W(\omega, t-1)R(\omega, X(t))$.

3 Investment Model

Wealth is generated through investment in the risky assets, but the trajectory of wealth can have large swings and the chance of falling below sustainable levels needs to be controlled. The characteristics of shortfalls (falling below benchmarks) are the rate/chance and the size. Both components are incorporated into our investment model, where shortfall rate is constrained at a specified level, and the shortfall size is penalized in the objective. The criterion in the objective is capital growth, namely the maximization of the expected value of logarithmic utility of penalized wealth, subject to the constraints.

3.1 Penalized Shortfall

We are concerned with trajectories which fall below a target path at discrete points in time. Consider a trajectory of the wealth process $W(t), t = 1, ..., T$, and the target/benchmark wealth path $w^*(t), t = 1, ..., T$. Two approaches to target paths are : (i) a growth path based on a desired growth rate, possibly at the risk free rate; (ii) a decay path based on a fallback rate. The targets are general and can vary from growth to decay over time.

If the trajectory is below the target at time t, $W(t) < w^*(t)$, then there is a penalty in the form of a wealth discount, $W(t)[1-\rho_t], \rho_t < 1$. Since the intention is to control large shortfalls, is is natural to make the penalty proportional to the shortfall, $\rho_t = \frac{w^*(t)-W(t)}{w^*(t)} = \frac{shortfall}{target}$. If $W(t) \geq w^*(t), \rho_t = 0$. Then discounted wealth is $W(t)\left[1-(1-\frac{W(t)}{w^*(t)})^+\right]^\gamma$, where the penalty parameter γ captures the decision makers aversion to losses and the positive part is defined by $[y]^+ = y$ if $y > 0$ and $[y]^+ = 0$ if $y \leq 0$. This discounting approach works well with a logarithmic transformation since when $W(t) < w^*(t)$

$$ln\left(W(t)\left[1-(1-\frac{W(t)}{w^*(t)})^+\right]^\gamma\right) = ln\left(W(t)\right) - \gamma\left[ln(w^*(t) - ln(W(t))\right]^+. \tag{10}$$

If $W(t) < w^*(t)$, the path *shortfall* is $[w^*(t) - W(t)]$ and the penalty $\gamma\left[ln(w^*(t)) - ln(W(t))\right]^+$ is *convex* in the shortfall. The penalty parameter $\gamma \geq 1$ is a power factor.

There is another rationale for the penalty approach. Consider $D(t) = \frac{W(t)}{w^*(t)}$. Then $ln(W(t)) = ln(w^*(t)) + ln(D(t)) = \left[0.5ln(w^*(t)) + ln(D^+(t))\right] + \left[0.5ln(w^*(t)) + ln(D^-(t))\right]$, where $D^+(t) = D(t)$ if $W(t) \geq w^*(t)$ and $D^+(t) = 1$ if $W(t) < w^*(t)$, $D^-(t) = D(t)$ if $W(t) < w^*(t)$ and $D^-(t) = 1$ if $W(t) \geq w^*(t)$. Then (9) can be written as

$$\left[0.5ln(w^*(t)) + ln(1 + \frac{[W(t) - w^*(t)]^+}{w^*(t)})\right].$$

$$+\left[0.5ln(w^*(t)) - ln((1 - \frac{[w^*(t) - W(t)]^+}{w^*(t)})^{-(1+\gamma)})\right]. \quad (11)$$

By separating the wealth into *gains* $[W(t) - w^*(t)]^+$and *losses* $[w^*(t) - W(t)]^+$ relative to the benchmark $w^*(t)$, it is seen that the convex penalty approach in (11) defines an objective which is concave in gains and convex in losses. So the convex penalty approach is consistent with one of the main principles of Prospect Theory (Kahneman and Tversky, 1993).

The use of the logarithmic transformation puts the focus on the growth rate of capital. That is, $ln(W(t)) = ln(W(t-1)) + ln(R(t))$, where $ln(R(t))$ is the rate of return in period t. In our model, the investor's objective is to achieve capital growth with security, so that the chance and size of shortfalls in wealth is small. This paper extends MacLean, Sangre, Zhao and Ziemba (2004), which constrained the chance of shortfall in a VaR model. The objective in that work was optimal growth, which was formulated as the maximization of the logarithm of terminal wealth, and it decomposed into the period by period growth rates. If wealth is discounted with a convex penalty in the objective as proposed, the same period by period decomposition applies.

To develop the wealth process and path shortfall, consider the return process $R(X(t)), t = 1, ..., T$. For the stochastic process $R(X(t))$ a trajectory of the data process is associated with an outcome ω in the space Ω of all returns trajectories, with probability space (Ω, B, P). Let $R(\omega, X(t)), \omega \in \Omega$, be a returns trajectory, where $X(t)$ is an investment strategy which can depend on the data history but not on unknown future returns. The wealth trajectory is $W(\omega, t) = W(\omega, t-1)R(\omega, X(t))$. A requirement that the wealth trajectory lies above the path is $W(\omega, t) \geq w^*(t), t = 1, ..., T$. For a set of trajectories $A \in B$, it could be required that all trajectories in the set satisfy the path condition: $W(\omega, t) \geq w^*(t), t = 1, ..., T, \omega \in A$.

In log space the corresponding path condition is $ln(W(\omega,t)) \geq ln(w^*(t)), t = 1,...,T, \omega \in A$. If the path constraint is not satisfied, the model imposes a penalty at the period of violation. That is, the logarithm of discounted wealth at the horizon is $ln(w(t_0)) + \sum_{t=1}^{T} ln(R(X(t)) - \gamma \sum_{t=1}^{T} [ln(w^*(t)) - ln(W(t)]^+$.

3.2 Capital Growth with Security

For the path condition to be satisfied $(1-\alpha)100\%$ of the time, the multiperiod capital growth problem, where the rate of shortfalls is controlled with a VaR constraint and the size is part of the objective, is written as

$$max \left\{ E \left[\sum_{t=1}^{T} ln(R(X(t)) - \gamma \sum_{t=1}^{T} [ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t))]^+ \right] \right\} \tag{12}$$

where

$$Pr[ln(R(X(t)) \geq ln(w^*(t)) - ln(W(t-1)), t = 1, \ldots, T] \geq 1 - \alpha \tag{13}$$

$$X^\top(t)e = 1, t = 1, ..., T.$$

For a decision X, the path condition is satisfied for a set of scenarios $A \in B$. If the measure of A is such that $P(A) \geq 1 - \alpha$, then the set is termed acceptable and the decision is feasible. There are potentially many feasible decisions for a specified acceptance set. Given an acceptance set $A = A_1 \times \ldots \times A_T, P(A) \geq 1 - \alpha$, with complement sets $\overline{A}_t$, a restricted form of the problem is

$$max \left\{ E \sum_{t=1}^{T} \left\{ ln(R(X(t)) - \gamma I_{\overline{A}_t} [ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t)))] \right\} \right\} \tag{14}$$

where

$$ln(R(\omega, X(t)) \geq ln(w^*(t)) - ln(W(\omega, t-1)), t = 1, \ldots, T, \omega \in A \tag{15}$$

$$X^\top(t)e = 1, t = 1, ..., T.$$

Let $\Psi(X^*(A))$ be the optimal solution for this problem. Then $sup_{\{A \in B, P(A) \geq 1-\alpha\}} \Psi(X^*(A)) = \Psi(X^*(A^*))$ is a solution to the full problem defined by (12),(13). It is assumed that there is an optimal solution to the full problem and therefore there is an optimal acceptance set. That is, given the optimal acceptance set the solutions for the alternative formulations are the same. The log transformation decomposes the final discounted wealth into a period by period summmation. The formulation with acceptance sets provides a setting for decomposing the multi-period constrained growth problem into a sequence of one period problems.

Proposition 1

Conditional on the optimal acceptance set, the optimal strategy in period t is path independent, depending on the wealth at the beginning of period t but not the path to that wealth. The problem is a sequence of static one period problems conditioned on the wealth from the previous period.

Proof:

Let A^* be the optimal acceptance set and consider the associated problem

$$max_X \left\{ E \sum_{t=1}^{T} \left\{ ln(R(X(t)) - \gamma I_{\overline{A}_t^*}[ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t)))] \right\} \right\}$$

Subject to

$$ln(R(\omega, X(t)) \geq ln(w^*(t)) - ln(W(\omega, t-1)), t = 1, \ldots, T, \omega \in A^*$$

$$X^\top(t)e = 1, t = 1, ..., T.$$

The Lagrangian for this problem is $L(X, \lambda^*, A^*) =$

$$E \sum_{t=1}^{T} \left[\left\{ ln(R(X(t)) - \gamma I_{\overline{A}_t^*}[ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t)))] \right\} \right] +$$

$$E \sum_{t=1}^{T} I_{A_t^*} \lambda_t^*(\omega) \left(ln(w^*(t)) - ln(W(t-1)) - ln(R(X(t))) \right). \tag{16}$$

The multiplier $\lambda_t^*(\omega) \geq 0$ is in the space of the Lesbegue integrable functions on Ω and is such that $max_X \{L(X, \lambda^*, A^*)\}$ is equivalent to the above problem. With $L_t(X, \lambda^*, A^*) =$

$$E \left[ln(R(X(t)) - \gamma I_{\overline{A}^*}[ln(w^*(t) - ln(W(t-1) - ln(R(X(t))] \right]$$

$$+ E \left[I_{A^*} \lambda_t(\omega) \left(ln(w^*(t)) - ln(W(t-1) - ln(R(\omega, X(t))) \right] \right], \tag{17}$$

then $L(X, \lambda^*, A^*) = \sum_{t=1}^{T} \{L_t(X, \lambda^*, A^*)\}$. It can be seen that the Lagrangian is a sequence of T expressions, each conditioned on the wealth outcome from the previous time period. That is, the decision $X(1)$ given the initial wealth $w(0)$ leads to wealth $W(1) = w(1)$, which is the wealth at the start of period $t = 2$. In the t^{th} period the Lagrangian maximization is equivalent to the one period problem

$$Max_{X(t)} \left\{ E \left[ln(R(X(t))) - \gamma I_{\overline{A}^*}[ln(w^*(t)) - ln(w(t-1)) - ln(R(X(t)))] \right] \right\} \tag{18}$$

Subject to

$$ln(R(X(t)) \geq (ln(w^*(t))) - ln(w(t-1)), \omega \in A^* \tag{19}$$

$$X(t)^\top e = 1.$$

That is, the dynamic multiperiod problem is a sequence of static one period problems conditional on the wealth from the previous period. □

The sequence of one period problems is defined for the optimal acceptance set A^*. Finding the optimal acceptance set is a difficult problem. There is a sequence of one period problems with a probability constraint which is equivalent to the optimal acceptance set sequence. So the probabilistic constraint contains the optimization over acceptance sets.

For the wealth process $ln(W(t)) = ln(W(t-1)) + ln(R(X(t))$, the constraint

$$Pr\left[ln(W(t-1)) + ln(R(X(t)) \geq ln(w^*(t))), t = 1, \ldots, T\right] \geq 1 - \alpha$$

is the same as

$$1 - Pr\left[\vee_{t=1}^{T}\left(ln(W(t-1)) + ln(R(X(t)) < ln(w^*(t))\right)\right] \geq 1 - \alpha.$$

.

With $Pr\left[ln(R(X(t)) < ln(w^*(t) - ln(w(t-1)\right] \leq \alpha_t$, where $\sum_{t=1}^{T} \alpha_t \leq \alpha$, the one period problem with a probabilistic constraint is

$$Max\left\{E\left[ln(R(X(t)) - \gamma[ln(w^*(t)) - ln(w(t-1)) - ln(R(X(t))]^+\right]\right\} \tag{20}$$

subject to

$$Pr\left[ln(R(X(t)) > ln(w^*(t) - ln(w(t-1)\right] \geq 1 - \alpha_t \tag{21}$$

$$X^\top(t)e = 1.$$

The requirement $\sum_{t=1}^{T} \alpha_t \leq \alpha$ is still part of the problem specification. The choice of α is replaced by a sequence $\{\alpha_t, t = 1, ..., T\}$. The α_t may be determined sequentially as the actual wealth trajectory $\{w(t), t = 1, ..., T\}$ unfolds relative to the benchmark path $\{w^*(t), t = 1, ..., T\}$. If a priori the periods are the same, the one period constraint probabilities would be $\alpha_t = \frac{1}{t}\alpha$. This is analogous to the Bonferroni method for determining an overall (path) rate α and period specific rates α_t.

3.3 Functional Form of One Period Problem

The multiperiod capital growth problem is structured as a linked sequence of one period problems, but the probabilistic constraints in the one period problem could pose a problem for solution. However, the setup for rates of return as normal within regimes and a mixture of normals overall makes the problem more tractable.

Assume that the distribution over regimes in period t is $(\pi_1(t), ..., \pi_m(t))$ and let the unconditional return be $R(X(t))$. The conditional rate of return given regime k in period t is multivariate normal with

$$ln(R_k(X(t))) = \left[X'(t)(\tilde{\phi}_k - \tilde{r}e) + r - \frac{1}{2}X'(t)\tilde{\Delta}_k^2 X(t)\right] + X'(t)\tilde{\Delta}_k Z_k.$$

If $ln(R_k(X(t))) < ln(w^*(t)) - ln(w(t-1))$, then $[ln(w^*(t) - ln(w(t-1) - ln(R_k(X(t))]^+$ has the same probability law as as $ln(R_k(X(t))$, which is Gaussian. Let $f_k(v|t), k = 1, ..., m$ be the normal density of $ln(R_k(X(t))$, the log-return given the regime is k. The unconditional distribution for log-returns is a mixture of normals $f(v|t) = \pi_1(t)f_1(v|t) + \ldots + \pi_m(t)f_m(v|t)$.

The chance constraint in the one period problem given $w(t-1)$, in terms of log-return, is

$$Pr\left[ln(R(X(t)) > ln(w^*(t)) - ln(w(t-1))\right] \geq 1 - \alpha_t,$$

or

$$\int_{-\infty}^{ln(w^*(t))-ln(w(t-1))} [\pi_1(t)f_1(v|t) + \ldots + \pi_m(t)f_m(v|t)]dv \leq \alpha_t. \tag{22}$$

Of course $\int_{-\infty}^{ln(w^*(t)-ln(w(t-1))} f_k(v|t)dv = \int_{-\infty}^{z_k^*(X(t))} f^\star(z)dz$, where $f^\star$ is the standard normal and $z_k^*(X(t)) = \frac{[ln(w^*(t))-ln(w(t-1))]-\tilde{\mu}_k(t)}{\tilde{\sigma}_k(t)}$, with $\tilde{\mu}_k(t) = X'(t)(\tilde{\phi}_k - \tilde{r}e) + \tilde{r} - \frac{1}{2}\tilde{\sigma}_k^2(t)$ and $\tilde{\sigma}_k^2(t) = X'(t)\tilde{\Delta}_k^2 X(t)$.

Let $G(X(t)) = \sum_{k=1}^{m} \pi_k(t) \int_{-\infty}^{z_k^*(X(t))} f^\star(z)dz - \alpha_t$. So the deterministic constraint is

$$G(X(t)) \leq 0. \tag{23}$$

The objective can be similarly reformulated. The expected rate of return is

$$F(X(t)) = E\left[ln(R(X(t)) - \gamma[ln(w^*(t)) - ln(w(t-1)) - ln(R(X(t)))]^+\right] =$$

$$\sum_{k=1}^{m} \pi_k(t) \cdot E(ln(R_k(X(t))-$$

$$\gamma \sum_{k=1}^{m} \pi_k(t) \int_{-\infty}^{z_k^*(X(t))} (ln(w^*(t)) - ln(w(t-1)) - [E(ln(R_k(X(t)) + z \cdot \sigma(ln(R_k(X(t)))]) f^\star(z)dz. \tag{24}$$

Then the one period problem is $P(g(t), \gamma, \alpha_t)$:

$$max\left\{F(X(t))|G(X(t)) \leq 0, X^\top(t)e = 1\right\}.$$

This problem depends on the gap between starting wealth and the path target, $g(t) = ln(w^*(t)) - ln(w(t-1)$, as well as the penalty γ and the shortfall probability α_t. The multi-period problem is a sequence of such one period problems, where the gap in the next period is controlled by the settings (γ, α_t) and the investment decisions for the period.

4 The Kelly Strategies

If the objective is to optimize capital growth, the problem is $Max\left\{ln(w_0) + \sum_{t=1}^{T} E(ln(R(X(t)))\right\}$. The expectation is over the randomness in prices and the uncertain regimes. The terminal wealth problem is a sequence of one period problems. In each period the optimal capital growth strategy, called the Kelly Strategy, maximizes the unconditional growth rate of capital $\sum_{k=1}^{m} \pi_k(t) \cdot E(ln(R_k(X(t)))$. This strategy has many attractive properties and has been dubbed *Fortunes Formula* (MacLean, Thorp and Ziemba (2010a), MacLean, Thorp, Zhao and Ziemba (2011)). One downside of the Kelly strategy is the chance of large losses. A prime motivation for the path constraint and shortfall penalty is to control large losses. In this section, the Kelly strategy and modifications are considered.

4.1 Kelly Strategy with Regimes

Consider the one period problem $max\sum_{k=1}^{m} \pi_k(t) \cdot E(ln(R_k(X(t))) = max\sum_{k=1}^{m} \pi_k(t) \cdot \left[\tilde{\mu}_k(t) - \frac{1}{2}\tilde{\sigma}_k^2(t)\right]$. Dropping the time argument the solution is

$$X^* = \left(\sum_{k=1}^{K} \pi_k \tilde{\Delta}_k^2\right)^{-1} \left(\sum_{k=1}^{K} \pi_k(\tilde{\phi}_k - \tilde{r}e)\right),$$

where $X^* = \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix}$ are the fractions invested in the n risky assets.

Since the distribution over regimes $(\pi_1, ..., \pi_K)$ in period t will depend on the regime distribution in period $t-1$ and the transition probabilities, the Kelly strategy will be prior regime dependent. It is noteworthy that the Kelly strategy does not depend on the wealth at the beginning of the period: $w(t-1)$. However, the *performance* of the Kelly strategy relative to the path does depend on the starting wealth. In the risk context, performance is defined by the *shortfall rate* and the average *shortfall size*.

Given starting wealth $w(t-1)$ and the position relative to the target path $g(t) = ln(w^*(t)) - ln(w(t-1)$, the *chance of a shortfall* in period t is $P[ln(R(X^*)) < g(t)] = \alpha_t^*$. Using the mixture of normals distribution,

$$\alpha_t^* = \int_{-\infty}^{g(t)} [\sum_{k=1}^{K} \pi_k f_k(v)] dv, \tag{25}$$

where $f_k(v)$ is the normal density with mean $\mu_k^* = Eln(R_k(X^*)) = X^{*'}\left(\tilde{\phi}_k - \tilde{r}e\right) + \tilde{r} - \frac{1}{2}X^{*'}\tilde{\Delta}_k^2 X^*$ and variance $\sigma_k^* = \sigma^2(ln(R_k(X^*)) = X^{*'}\tilde{\Delta}_k^2 X^*$.

The *average size of a shortfall* with the Kelly strategy, is

$$\eta_t^* = \frac{1}{\alpha_t^*}\left\{\int_{-\infty}^{g(t)} \pi_k [\sum_{k=1}^{K} (g(t) - v) f_k(v)] dv\right\}. \tag{26}$$

where $V_k = ln(R_k(X^*)$. The bi-criteria (α_t^*, η_t^*) can be combined into the ***risk score***

$$\varphi_t^* = \alpha_t^* \times \eta^*,$$

describing the risk relative to the path benchmark $w^*(t)$ and starting wealth $w(t-1)$.

The formulas for the rate and size of shortfalls can be calculated to determine the risk with the Kelly strategy. Obviously that risk will depend on the characteristics of the financial market $\Theta = (\pi(0), P, \theta_1, \ldots, \theta_m)$, and the investors financial position $g(t)$.

CASE : Single Risky Asset

To simplify the analysis of shortfall rate and shortfall size with the Kelly strategy, consider the case of a single risky asset (the market index) and two regimes representing *UP* and *DOWN* markets. It is assumed the probabilities for *UP*/*DOWN* in the coming period are (π_1, π_2). So the model parameters are $\Theta = (\tilde{\phi}_1, \tilde{\delta}_1^2, \pi_1, \tilde{\phi}_2, \tilde{\delta}_2^2, \pi_2)$. If $\bar{\phi} = \pi_1\tilde{\phi}_1 + \pi_2\tilde{\phi}_2$, and $\bar{\delta}^2 = \pi_1\tilde{\delta}_1^2 + \pi_2\tilde{\delta}_2^2$, the Kelly strategy invests $x^* = \frac{\bar{\phi}-r}{\bar{\delta}^2}$ in the risky asset.

From (25) and (26) the *shortfall rate* and average *shortfall size* are respectively,

$$\alpha_t^* = \pi_1\Phi(z_1^*) + \pi_2\Phi(z_2^*) \tag{27}$$

$$\eta_t^* = g(t) - \frac{1}{\alpha_t^*}\left\{\pi_1\left[\mu_1^*\Phi(z_1^*) - \sigma_1^*\Phi'(z_1^*)\right] + \pi_2\left[\mu_2^*\Phi(z_2^*) - \sigma_2^*\Phi'(z_2^*)\right]\right\}, \tag{28}$$

where Φ is the standard normal cumulative distribution and

$$z_1^* = \frac{g(t) - \left[x^*(\tilde{\phi}_1 - \tilde{r}) + \tilde{r} - \frac{1}{2}x^{*2}\tilde{\delta}_1^2\right]}{x^*\tilde{\delta}_1}$$

$$z_2^* = \frac{g(t) - \left[x^*(\tilde{\phi}_2 - \tilde{r}) + \tilde{r} - \frac{1}{2}x^{*2}\tilde{\delta}_2^2\right]}{x^*\tilde{\delta}_2}.$$

.

The expressions for α_t^* and η_t^* are defined by the wealth relative to the benchmark as given by g as well as the standard normal distribution Φ, the mean μ_k^* and standard deviation σ_k^* of the return on the Kelly investment strategy in each regime, and the regime probabilities (π_1, π_2) . The Kelly strategy and investment returns depend on the price parameters $\left(\tilde{\phi}_1, \tilde{\delta}_1, \tilde{\phi}_2, \tilde{\delta}_2\right)$. Let $\tilde{r} = 0, \tilde{\phi}_1 = \phi, \tilde{\phi}_2 = (1-c)\phi_1, \tilde{\delta}_1 = \tilde{\delta}_2 = \delta, \pi_1 = 1-\pi, \pi_2 = \pi$. The constant c defines the DOWN returns relative to the UP returns and is a factor in price volatility.

PROPOSITION 2

The Kelly investment in the risky asset decreases as c increases. Let (α^*, η^*) *are the* *(shortfall rate, expected shortfall size) for the Kelly strategy. If the wealth gap is g, then there exists a value g^*such that: (i) the rate of shortfall α^* and the expected shortfall size η^* increase as the downside return parameter c increases when $g > g^*$; (ii) the rate of shortfall α^* and the expected shortfall size η^* decrease as the downside return parameter c increases when $g < g^*$.*

Proof:

The Kelly strategy is $x^* = (1 - c\pi)\frac{\phi}{\delta^2.}$ and $\frac{\partial x^*}{\partial c} = -\pi\frac{\phi}{\delta^2} < 0$.

With $\alpha^* = (1-\pi)\,\Phi(z_1^*) + \pi\Phi(z_2^*)$, then $\frac{\partial \alpha^*}{\partial c} = (1-\pi)\Phi'(z_1^*)\frac{\partial z_1^*}{\partial c} + \pi\Phi'(z_2^*)\frac{\partial z_2^*}{\partial c}$. With $z_2^* = z_1^* + \frac{\phi}{\delta}c$, simple algebra gives $\frac{\partial z_1^*}{\partial c} = \frac{\pi\delta}{\phi(1-c\pi)^2}\left[g - \frac{0.5\phi^2}{\delta^2}(1-c\pi)\right], \frac{\partial z_2^*}{\partial c} = \frac{\partial z_1^*}{\partial c} + \frac{\phi}{\delta}$. $\frac{\partial z_1^*}{\partial c}$ is monotone increasing in g and is negative when $g < \frac{0.5\phi^2}{\delta^2}(1-c\pi)$ and positive when $g > \frac{0.5\phi^2}{\delta^2}(1-c\pi)$. There is a value g^* such that $\frac{\partial \alpha^*}{\partial c} < 0$ if $g < g^*$and $\frac{\partial \alpha^*}{\partial c} > 0$ if $g > g^*$.

The expected shortfall size is $\eta^* = g - \frac{H(c)}{\alpha^*}$, where

$H(c) = (1-\pi)\left[\mu_1^*\Phi(z_1^*) - \sigma_1^*\Phi'(z_2^*)\right] + \pi\left[\mu_2^*\Phi(z_2^*) - \sigma_2^*\Phi'(z_2^*)\right]$. So$\frac{\partial \eta^*}{\partial c} = \frac{\partial \alpha^*}{\partial c} \times \frac{\partial \eta^*}{\partial \alpha^*}$. Since $\frac{\partial \eta^*}{\partial \alpha^*} = \frac{H(c)}{\alpha*^2} > 0$, the increase/decrease in expected shortfall size behaves the same as the increase/decrease in shortfall rate. □

The chance of falling below the path and the expected size of the shortfall depend on the starting position as given by the gap g. Recall that $g(t) = ln(w^*(t) - ln(w(t-1))$, so that $g(t) < 0$ is the case where the target benchmark is below the starting wealth and $g(t) > 0$ when the target is above the starting wealth. The fact that the performance of the Kelly strategy depends on wealth suggests that current wealth should be a factor in investment decisions.

Example 1 The qualitative results consider the effect on downside risk of the expected rate of return in the down regime and the wealth gap. The effect will be illustrated with an example. In the computations the values $\pi_1 = 0.8, \pi_2 = 0.2, \tilde{r} = 0.00004, \tilde{\phi}_1 = .0003, \tilde{\delta}_1^2 = .00015, \tilde{\delta}_2^2 = .00015$ will be used, based on one day returns. Consider, then, in Table 1 the *shortfall rate* and in Table 2 *average shortfall size* for a range of risky investment scenarios.

c	$\tilde{\phi}_1$	$\tilde{\phi}_2$	x^*	g				
				-0.002	**-0.006**	**-0.010**	**-0.014**	**-0.018**
2.0	.0003	-.0003	.93	.43	.30	.19	.11	.06
2.2	.0003	-.00036	.85	.42	.28	.17	.09	.04
2.4	.0003	-.00042	.77	.41	.26	.14	.07	.03
2.6	.0003	-.00048	.69	.40	.24	.12	.05	.02
2.8	.0003	-.00054	.61	.39	.21	.09	.03	.01
3.0	.0003	-.00060	.53	.37	.18	.06	.02	.00
3,2	.0003	-.00066	.45	.35	.14	.04	.01	.00
3.4	.0003	-.00072	.37	.32	.09	.01	.00	.00
3.6	.0003	-.00078	.29	.28	.05	.00	.00	.00
3.8	.0003	-.00084	.21	.21	.01	.00	.00	.00

Table 1: Shortfall Rate: Kelly Strategy

The scenarios are defined by the relative rates in UP and DOWN regimes $c = 1 - \frac{\tilde{\phi}_2}{\tilde{\phi}_1}, \tilde{\phi}_1 > 0, \tilde{\phi}_2 < 0$, and the gap $g = ln(w^*) - ln(w)$. In the table the gap is shown as negative, that is the wealth at the beginning of the period is above the target. A shortfall occurs if the one period return is less than the gap.

The shortfall rate drops dramatically if the starting position is favorable relative to the path target. As the downside decreases the Kelly fraction also drops since the investment is less favorable. The most favorable market scenario ($c = 2$) has an annual return of 4.8% and the least favorable ($c = 3.8$) has an annual return of 1.8%. The kelly strategy is aggressive and risky in favorable markets.

The shortfall size as the regime parameter c and the gap above the path g change is provided in Table 3 for the same settings as Table 2. These numbers show the average size of the shortfall in terms of daily rate of returns, so the shortfalls are substantial. For example in the situation where beginning wealth is close to the target ($g = -0.002$), we have for shortfalls $\frac{W}{w^*} = 0.993$ or 99.3% of the target wealth on average. This level of fallback is equivalent to 17% of starting wealth on an annualized basis. The pattern in average size is similar to that for the shortfall rate , with the relative returns in UP and DOWN markets having a slight negative effect.

c	g				
	-0.002	**-0.006**	**-0.010**	**-0.014**	**-0.018**
2.0	0.0084	0.0072	0.0063	0.0055	0.0049
2.2	0.0076	0.0065	0.0056	0.0048	0.0042
2.4	0.0068	0.0057	0.0049	0.0042	0.0036
2.6	0.0061	0.0050	0.0042	0.0035	0.0031
2.8	0.0053	0.0043	0.0035	0.0029	0.0025
3.0	0.0045	0.0035	0.0028	0.0023	0.0020
3,2	0.0038	0.0028	0.0022	0.0018	0.0015
3.4	0.0030	0.0021	0.0016	0.0013	0.0010
3.6	0.0022	0.0015	0.0011	0.0008	0.0007
3.8	0.0015	0.0009	0.0006	0.0005	0.0004

Table 2: Shortfall Size: Kelly Strategy

For the scenarios presented in Table 1 and Table 2 the Kelly strategy gets more conservative as the downside decreases, since the upside is constant. More volatile scenarios with increasing upside to match decreasing downside would keep the Kelly investment proportion high with a corresponding high risk in terms of the rate and size of shortfalls.

4.2 Penalizing Shortfalls in the One Period Problem

The Kelly strategy can have an unacceptable risk of shortfalls particularly in favorable markets, and that is the motivation for controlling the rate and size of shortfalls relative to a benchmark. An intuitive approach is to use a fractional Kelly strategy, where the proportional allocation to risky assets is the same as the Kelly but the total wealth invested in risky assets is reduced to a fraction of the Kelly investment in risky assets. There are a variety of ways for determining the fraction, including using a power utility. (MacLean, Zhao and Ziemba, 2006.) The fraction will be considered here from the perspective of the VaR constraint and path penalty. There is a single risky asset (Kelly portfolio), and the investment fraction in the Kelly portfolio is selected to satisfy both the rate constraint and the size penalty. The strategy will depend on the path and the starting wealth, as opposed to the pure Kelly.

The constrained one period problem problem is

$$max\left\{F(X(t))|G(X(t))\leq 0, X^{\top}(t)e=1\right\}.$$

Consider the market described previously with UP/DOWN regimes, where $\tilde{\phi}_1=\phi,\tilde{\phi}_2=(1-c)\phi,\tilde{\delta}_1=\tilde{\delta}_2=\delta,\pi_1=1-\pi,\pi_2=\pi$. With x the fraction invested in the risky asset (Kelly portfolio), let $x=fx^*$, where $x^*=argmax\left\{\left((1-c\pi)\phi x-\frac{1}{2}x^2\delta^2\right)\right\}$ is the Kelly strategy. The Kelly shortfall rate is $\alpha^*=Pr\left[ln(R(x^*)<g\right]$. Assume $x^*>0$.

Proposition 3

Let f^*be the optimal fraction for the constrained one period problem, the required shortfall rate be α and the shortfall size penalty be γ.

(i) For given penalty, the optimal fraction f^*decreases as the rate α decreases.

(ii) For given rate, the optimal fraction f^*decreases as the penalty γ increases.

Proof:

Consider $F(x)=\left((1-c\pi)\phi x-\frac{1}{2}x^2\delta^2\right)-\gamma g(t)\left(\pi_1\Phi(z_1)+\pi_2\Phi(z_2)\right)$ $+\gamma\left\{\pi_1\left[\mu_1\Phi(z_1)-\sigma_1\Phi'(z_1)\right]+\pi_2\left[\mu_2\Phi(z_2)-\sigma_2\Phi'(z_2)\right]\right\}$ and $G(x)=\pi_1\Phi(z_1)+\pi_2\Phi(z_2)-\alpha$, where $\mu_k=x\left(\tilde{\phi}_k-\tilde{r}\right)+\tilde{r}-\frac{1}{2}x^2\tilde{\delta}_k^2$ and $\sigma_k=x\tilde{\delta}_k$, and $z_k=\frac{g(t)-\left[x(\tilde{\phi}_k-\tilde{r})+\tilde{r}-\frac{1}{2}x^2\tilde{\delta}_k^2\right]}{x\tilde{\delta}_k}$ for $k=1,2$.

The following inequalities hold for $0<x<x^*$,

$$\frac{\partial}{\partial x}\left((1-c\pi)\phi x-\frac{1}{2}x^2\tilde{\delta}^2\right)>0$$

$$\frac{\partial}{\partial x}\left(\pi_1\Phi(z_1)+\pi_2\Phi(z_2)\right)>0$$

$$\frac{\partial}{\partial x}\left\{\pi_1\left[\mu_1\Phi(z_1)-\sigma_1\Phi'(z_1)\right]+\pi_2\left[\mu_2\Phi(z_2)-\sigma_2\Phi'(z_2)\right]\right\}<0$$

$$\frac{\partial}{\partial x}\left\{\pi_1\left[\mu_1\Phi(z_1)-\sigma_1\Phi'(z_1)\right]+\pi_2\left[\mu_2\Phi(z_2)-\sigma_2\Phi'(z_2)\right]-g(t)\left(\pi_1\Phi(z_1)+\pi_2\Phi(z_2)\right)\right\}<0.$$

Consider the Lagrangian $L(x,\lambda)=F(x)-\lambda G(x)$. Let $L(x,\lambda|\alpha,\gamma)=\left((1-c\pi)\phi x-\frac{1}{2}x^2\delta^2\right)$ $+\gamma\left\{\pi_1\left[\mu_1\Phi(z_1)-\sigma_1\Phi'(z_1)\right]+\pi_2\left[\mu_2\Phi(z_2)-\sigma_2\Phi'(z_2)\right]-g(t)\left(\pi_1\Phi(z_1)+\pi_2\Phi(z_2)\right)\right\}$ $-\lambda[(\pi_1\Phi(z_1)+\pi_2\Phi(z_2))-\alpha]$. For optimal multiplier λ^*, $max_x\left\{L(x,\lambda^*|\alpha,\gamma_1)\right\}\geq$ $max_x\left\{L(x,\lambda^*|\alpha,\gamma_2)\right\}$ for $0<\gamma_1<\gamma_2$ and $x_{\gamma_1}=argmax\left\{L(x,\lambda^*|\alpha,\gamma_1)\right\}\geq x_{\gamma_2}=argmax\left\{L(x,\lambda^*|\alpha,\gamma_2)\right\}$.

With $x_{\gamma_1}=f_{\gamma_1}x^*, x_{\gamma_2}=f_{\gamma_2}x^*$, we have $f_{\gamma_1}\geq f_{\gamma_2}$. Similarly $max_x\left\{L(x,\lambda^*|\alpha_1,\gamma)\right\}\leq max_x\left\{L(x,\lambda^*|\alpha_2,\gamma)\right\}$ for $0<\alpha_1<\alpha_2$ implies $f_{\alpha_1}\leq f_{\alpha_2}$.□

Example 2

Consider the one period problem with starting wealth gap g and reliability level $\alpha=0.05$, where $\pi_1=0.8,\pi_2=0.2,\tilde{r}=0.00015$ and the values for daily return on stocks are $\tilde{\phi}_1=0.000375,\tilde{\phi}_2=-0.0005$ and $\tilde{\delta}_1^2=\tilde{\delta}_2^2=0.000225$. For this example the Kelly strategy is to invest the fraction $x^*=0.22$ in stock. Recall that $g=w^*(t)-w(t-1)$, the wealth position relative to the path benchmark. Table 4 gives the investment in stock as a fraction of the Kelly for a range of values for the parameter g and penalty parameter γ.

	g							
γ	**-.016**	**-.014**	**-.012**	**-.010**	**-.008**	**-.006**	**-.004**	**-.002**
0	1.0	1.0	0.99	0.91	0.78	0.57	0.50	0.49
2	1.0	0.98	0.92	0.81	0.67	0.52	0.36	0.19
10	1.0	0.95	0.86	0.74	0.61	0.47	0.32	0.17

Table 3: Kelly Fractions: f^*

If starting wealth is close to the path target, the one period investment strategy is conservative when the shortfall rate constraint is imposed. A higher starting position leads to investment closer to the full Kelly strategy. In the growth framework it is never optimal to invest more than the full Kelly because growth falls and risk increases. The effect of the penalty on the investment fraction is significant, with the fraction decreasing as the penalty increases.

5 Multiperiod Multi-asset Problem

The capital growth problem with path shortfall conditions is a multiperiod problem which is decomposed into a sequence of one period problems. The analysis of the Kelly strategy in Section 4 considered the one period problem, and restricted investment opportunities to a single asset, the Kelly portfolio. For the problem with a VaR constraint and penalized objective, it is clear that introducing the path through the gap parameter g has a substantial impact on the single period investment strategy. In this section the problem with multiple periods and multiple risky assets is considered. When there are many risky assets in the constrained growth problem, the proportional investments in assets would usually not correspond to the Kelly portfolio. The gap is the linking condition between periods in the multiperiod problem. To observe the pattern in the gap between current wealth and the path target and its effect on the sequence of investment decisions in multiple assets, a multiperiod problem is solved as a linked sequence of one period multi-asset problems.

Assume there is a risk free asset and two risky assets, stocks and bonds, with the investment fractions in period t being $(x_0(t), x_1(t), x_2(t))$. There is initial wealth of $w(0) = \$100$ and three regimes in the financial market; bull, transition, bear. The regimes are driven by a Markov switching process. The risky returns are considered to be lognormal, with parameter settings for daily asset price dynamics as follows:

- Daily rate of return on stocks and bonds $\tilde{\phi}_j, j = 1, 2, 3$, respectively (annualized):

$$\begin{pmatrix} -0.1 \\ 0.035 \end{pmatrix}, \begin{pmatrix} 0.015 \\ 0.06 \end{pmatrix}, \begin{pmatrix} 0.25 \\ -0.02 \end{pmatrix}.$$

- Covariance of daily returns$\tilde{\Delta}_j^2$ by regime $j = 1, 2, 3$,respectively (annualized):

$$\begin{pmatrix} 0.09 & -0.0108 \\ -0.0108 & 0.0324 \end{pmatrix}, \begin{pmatrix} 0.0484 & 0.0099 \\ 0.0099 & 0.0225 \end{pmatrix}, \begin{pmatrix} 0.0025 & -0.0015 \\ -0.0015 & 0.01 \end{pmatrix}.$$

- DailyTransition matrix for regimes:

$$P = \begin{pmatrix} 0.75 & 0.15 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{pmatrix}.$$

- Initial probability for regimes:

$$\pi(0) = (0, 1, 0)\,.$$

- Daily Risk free rate (annualized):

$$\tilde{r} = 0.02.$$

- The target path will be developed sequentially depending on the status with respect to the path in the previous period. If there is a shortfall, so that $w(t-1) < w^*(t-1)$, then $w^*(t) = 1.01 \times w(t-1)$. If the target is exceeded , so that $w(t-1) > w^*(t-1)$, then $w^*(t) = 0.99 \times w(t-1)$. The intention is to relax the path requirement if the results are postive, thereby taking more risk. If the realized return is below the target the investor is constrained to recover some of the loss and that will force a more conservative strategy. The chosen rates $(0.99, 1.01)$ are for illustration.

The investment decisions are made daily for a period of one year: $T = 256$. At each time point the one period problem with the VaR constraint and penalty on path violations is

$$max_{X(t)} \left\{ F(X(t), \gamma, g(t)) | G(X(t), \alpha, g(t)) \leq 0, X^{\top}(t)e = 1 \right\},$$

where $F(X(t), \gamma, g(t)) =$

$$\sum_{k=1}^{3} \pi_k(t) \cdot [\mu_k(t)] -$$

$$\gamma \sum_{k=1}^{3} \pi_k(t) \int_{-\infty}^{z_k^*(X(t))} (g(t) - [\mu_k(t) + z \cdot \sigma_k(t)]) f^\star(z) dz,$$

and $G(X(t), \gamma, g(t)) = \sum_{k=1}^{3} \pi_k(t) \int_{-\infty}^{z_k^*(X(t))} f^\star(z) dz - \alpha_t$.

This is a nonconvex problem and a Monte Carlo approach is used to get the solution. (Mockus, 1989.)

In the one period problem at the start of period t , the values $(\pi_1(t), \ldots, \pi_3(t))$ are determined by the transition probabilities and the distribution over the regimes at time $t-1$. The transition probabilities and the asset return parameters are known in the problem studied here. In practice the parameters and probabilities are estimated. The EM algorithm (Dempster, 1977; Hamilton, 1989) is used to estimate those values in the hidden Markov model from the history of returns on risky assets.

To test the decision model, 5000 trajectories of 256 trading days are generated from the returns distributions. Along a trajectory the single period decision algorithm is implemented sequentially. A variety of performance statistics are calculated:

1. The violation probability = the relative frequency with which the wealth at the end of a period drops below the path target for that period. That is an average over the multiple decision periods.

2. The average final wealth = the average over the 5000 trajectories of the wealth after 256 trading days.

3. The Sharpe ratio for final wealth = the average risk adjusted return divided by the standard deviation.

4. The average cumulative shortfall = the average over the 5000 trajectories of the total shortfall along a trajectory.

Performance statistics for a selection of risk control settings are presented in Table 4.

Measure	Alpha	Gamma		
		0	2	10
Violation Probability	0.05	0.1482	0.0183	0.0040
	0.25	0.3504	0.0186	0.0037
	0.50	0.4100	0.0154	0.0040
Av Final Wealth	0.05	113.0546	106.2743	105.1149
	0.25	118.8159	106.5073	105.3578
	0.50	121.4290	106.2784	105.0543
Sharpe Ratio	0.05	0.5229	0.5558	0.5862
	0.25	0.5156	0.5328	0.6590
	0.50	0.5408	0.5866	0.5940
Av Cum shortfall	0.05	-70.7486	-7.9375	-2.0899
	0.25	-177.6402	-8.3683	-1.7586
	0.50	-243.5916	-5.6936	-1.7966

Table 4: Performance Statistics

The $\gamma = 0$ numbers are for the investment model with the path VaR constraint, but no penalty on shortfalls. The weak constraint with $\alpha = 0.50$ has a lot of downside. This corresponds to the Kelly case,where there is maximum growth rate but potentially large downside losses. Even the stricter condition with $\alpha = 0.05$ still has a high shortfall rate and high average cumulative shortfall when there is no penalty in the objective.

When the penalty on shortfalls is introduced, the effect is quite dramatic. Both the rate and average size are decreased substantially. The largest penalty $\gamma = 10$ has negligible rate and average size, and the average final wealth is comparable to that of the moderate penalty of $\gamma = 2$.

To indicate the effect of the risk control settings (α, γ) on decisions the monthly average weights in stocks, bonds and cash are given in Figure 1 for values of α and γ.

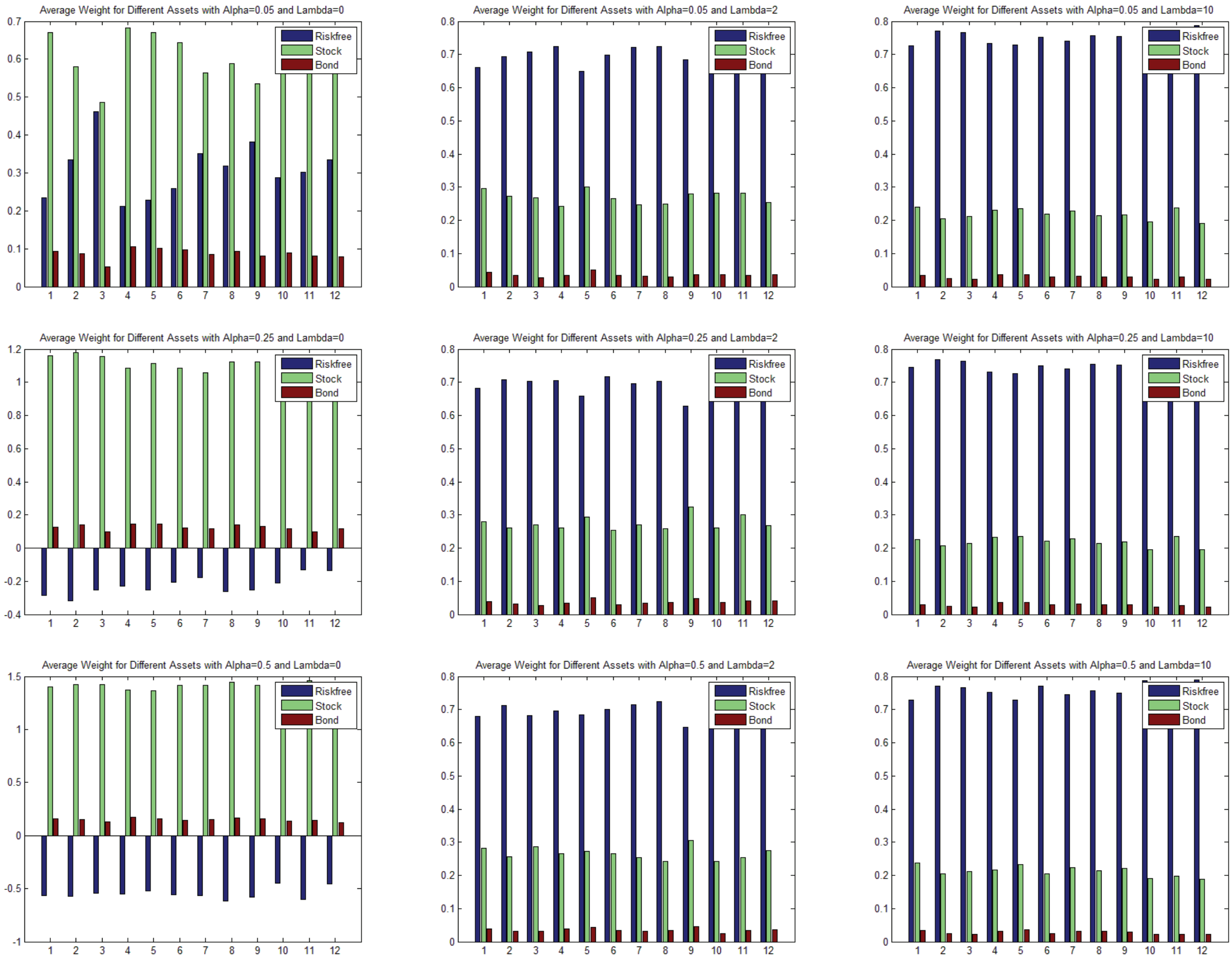

Figure 1: Investment Fractions

The Kelly case where $\alpha = 0.5, \gamma = 0$ is highly levered, with strong investment in stock financed by borrowing - negative in the risk free asset. Without the penalty ($\gamma = 0$), the VaR constraint is active when $\alpha = 0.05$, and the effect is a dramatic shift to the risk free asset. The investment fractions in the stock and bond are different from the Kelly, that is the strategy is not fractional Kelly. When the penalty is introduced into the objective the fraction in stocks declines substantially for all α settings, with a corresponding increase in the fraction in the risk free asset. The total investment in risky assets changes with the control settings, and also the relative fraction of the risky investment in stocks and bonds changes. The solution in the multiple risky asset case is not fractional Kelly per se, but it is an optimal growth strategy subject to control conditions.

An important goal of the VaR constraint and shortfall penalty is to smooth the wealth trajectory. Figure 2 displays a sampling of 10 trajectories for values of α and γ. A lower bound of 96 and an upper bound of 116 are used to emphasize effects. Again the effect of the penalty is significant. The downside is controlled and the majority of trajectories experience positive growth at each time period.

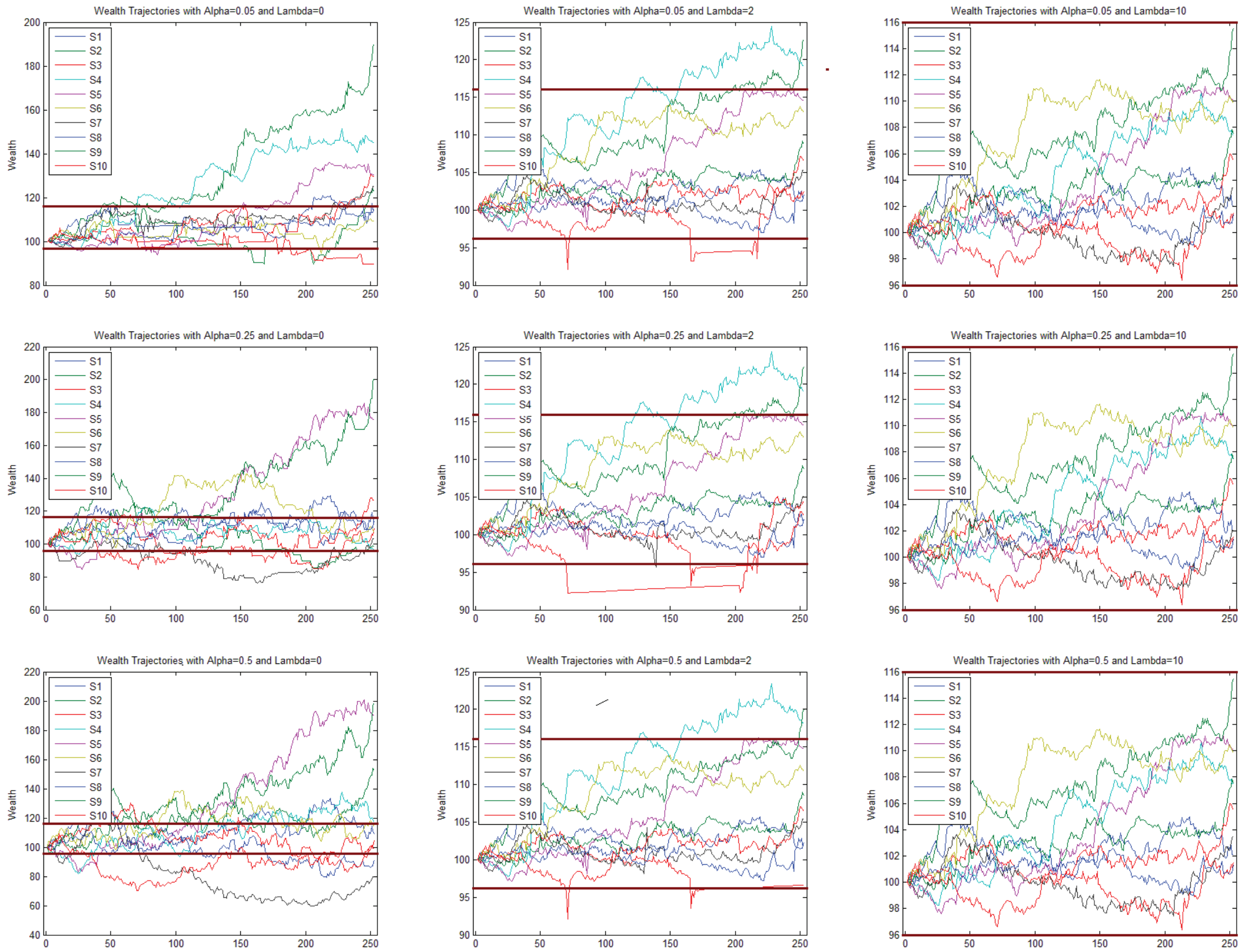

Figure 2: Trajectories

6 Conclusion

In the Kelly optimal capital growth problem where the rate of return to the horizon is maximized, the solution is usually very aggressive and the chance of significant loss of capital in the short to medium term is too large. A VaR constraint on the wealth trajectory controls the risk of losses, but the size of losses is crucial. In this paper both the chance and size of losses is controlled. The loss shortfall is penalized in the objective with a wealth discounting approach. This retains the geometric character of the wealth process, or equivalently the arithmetic character of log-wealth. The model parameters are the VaR level, the VaR probability α, and the shortfall penalty γ. The impact of the parameters on strategies and accumulated capital is studied analytically with one risky asset (security) and a riskless asset. The methodology is also applied to the fundamental problem of investing in stocks and bonds over time. The convex penalty has the advantage of smoothing the trajectory of acculated capital while achieving capital growth. Excessive penalization of shortfalls leads to a path with little volatility, but it falls below low penalty paths along the full trajectory.

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