A Fixed Point Theorem for Measurable Selection Valued Correspondences Induced by Upper Caratheodory Correspondences

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Abstract
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Keywords: m-tuples of Caratheodory functions, upper Caratheodory correspondences, continuum valued upper Caratheodory sub-correspondences, weak star upper semicontinuous measurable selection valued correspondences, approximate Caratheodory selections, fixed points of nonconvex, measurable selection valued correspondences induced by the composition of an m-tuple of Caratheodory functions with a continuum valued upper Caratheodory sub-correspondence.

JEL Classification: C7

AMS Classification (2010): 28B20, 47J22, 55M20, 58C06, 91A44

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Abstract

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Key Words: m-tuples of Caratheodory functions, upper Caratheodory correspondences, continuum valued upper Caratheodory sub-correspondences, weak star upper semicontinuous measurable selection valued correspondences, approximate Caratheodory selections, fixed points of nonconvex, measurable selection valued correspondences induced by the composition of an m-tuple of Caratheodory functions with a continuum valued upper Caratheodory sub-correspondence.

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1 Introduction

We show that any measurable selection valued correspondence induced by the composition of an m-tuple of real-valued Caratheodory functions with an upper Caratheodory (uC) correspondence has fixed points if the underlying uC correspondence in the composition contains a continuum valued uC sub-correspondence. Moreover, we show that the induced composition sub-correspondence is upper semicontinuous in the appropriate weak star topologies.

2 Primitives, Assumptions, and Preview

Let (Ω, B_Ω, μ) be a probability space where Ω is a complete, separable metric space with metric ρ_Ω, B_Ω the Borel σ-field generated by the ρ_Ω-open sets in Ω, and μ a regular Borel probability measure. Let $Y := [-M, M]^m \subset R^m$ where $M > 0$ and let $X := X_1 \times \cdots \times X_m$ where for each $d = 1, 2, \ldots, m$, X_d is a convex, compact metrizable subset of a locally convex Hausdorff topological vector space E_d equipped with a metric ρ_{X_d} compatible with the locally convex topology inherited from E_d. Finally, equip Y with sum of absolute values metric, $\rho_Y(y, y') := \sum_d \rho_{Y_d}(y_d, y'_d) := \sum_d |y_d - y'_d|$ and equip X with the sum metric, $\rho_X := \sum_d \rho_{X_d}$, compatible the product topology inherited from $E = E_1 \times \cdots \times E_m$. Next, let $\mathcal{L}^\infty_Y := \mathcal{L}^\infty_{Y_1} \times \cdots \times \mathcal{L}^\infty_{Y_m}$, where for each $d = 1, 2, \ldots, m$, $\mathcal{L}^\infty_{Y_d}$ is a convex, weak star compact metrizable subset of $\mathcal{L}^\infty_R$, the Banach space of μ-equivalence classes of μ-essentially bounded, measurable, real-valued functions, where $v \in \mathcal{L}^\infty_Y$ if and only if $v(\omega) := (v_1(\omega), \ldots, v_m(\omega)) \in Y$ a.e. $[\mu]$. Equip $\mathcal{L}^\infty_Y$ with the sum metric, $\rho_{w^*} := \sum_d \rho_{w^*_d}$, compatible the weak star product topology inherited from $\mathcal{L}^\infty_{R^m}$. Finally, let $P_f(X)$ be the hyperspace of nonempty ρ_X-closed subsets of X.

Consider an *upper Caratheodory* (uC) correspondence,

$$\mathcal{N}(\cdot, \cdot) : \Omega \times \mathcal{L}^\infty_Y \longrightarrow P_f(X), \tag{1}$$

jointly measurable in (ω, v) and upper semicontinuous in v for each ω. We call the collection of upper semicontinuous correspondences, $\{\mathcal{N}(\omega, \cdot) : \omega \in \Omega\}$ the USCO part (Hola and Holy, 2015), and $\{\mathcal{N}(\cdot, v) : v \in \mathcal{L}^\infty_Y\}$ the measurable part of the uC correspondence $\mathcal{N}$. Denote by $\mathcal{UC}_{\Omega \times \mathcal{L}^\infty_Y\text{-}P_f(X)}$ the collection of all such uC correspondences.

Next consider the Y-valued Caratheodory function,

$$(\omega, v, x) \longrightarrow u(\omega, v, x) := (u_1(\omega, v, x), \ldots, u_m(\omega, v, x)) \in Y, \tag{2}$$

measurable in ω and jointly continuous in (v, x), and let

$$\mathcal{P}(\cdot, \cdot) : \Omega \times \mathcal{L}^\infty_Y \longrightarrow P_f(Y), \tag{3}$$

denote the composition of uC correspondence $\mathcal{N}(\cdot, \cdot)$ with the m-tuple of Caratheodory functions, $(u_1(\cdot, \cdot, \cdot), \ldots, u_m(\cdot, \cdot, \cdot))$. For each $(\omega, v) \in \Omega \times \mathcal{L}^\infty_Y$ let

$$\mathcal{P}(\omega, v) := u(\omega, v, \mathcal{N}(\omega, v)). \tag{4}$$

The correspondence, $\mathcal{P}(\cdot, \cdot)$, is also a uC correspondence. We will call such a correspondence a uC composition correspondence.

Each uC composition correspondence, $\mathcal{P}(\cdot, \cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}^\infty_Y\text{-}P_f(Y)}$, induces a measurable selection valued correspondence,

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v)) := \mathcal{S}^\infty(\mathcal{P}_v), \tag{5}$$

where for each $v \in \mathcal{L}_Y^\infty$, $\mathcal{S}^\infty(\mathcal{P}_v)$ is the collection of μ-equivalence classes of functions u in $\mathcal{L}_Y^\infty$ such that $u(\omega) \in \mathcal{P}(\omega, v)$ a.e. $[\mu]$. We will show that for all such uC composition correspondences,

$$\left.\begin{array}{c} v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v)) = \mathcal{S}^\infty(u(\cdot, v, \mathcal{N}(\cdot, v))) \\ = (\mathcal{S}^\infty(u_1(\cdot, v, \mathcal{N}(\cdot, v)), \ldots, \mathcal{S}^\infty(u_m(\cdot, v, \mathcal{N}(\cdot, v))), \end{array}\right\} \tag{6}$$

if the underlying uC correspondence, $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}P_f(X)}$, contains a *continuum valued sub-correspondence*, $\eta(\cdot,\cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}P_f(X)}$ (i.e., a uC correspondence $\eta(\cdot,\cdot)$ taking continuum values such that $Gr\eta(\omega,\cdot) \subset Gr\mathcal{N}(\omega,\cdot)$ for all ω) then its uC composition correspondence, $(\omega, v) \longrightarrow u(\omega, v, \eta(\omega, v))$, induces a selection sub-correspondence,

$$v \longrightarrow \mathcal{S}^\infty(p(\cdot, v)) := \mathcal{S}^\infty(u(\cdot, v, \eta(\cdot, v))), \tag{7}$$

that is weak star upper semicontinuous and has fixed points. Thus while the original selection correspondence, $v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v)$, may fail to be weak star upper semicontinuous, its selection sub-correspondence induced by a continuum valued uC sub-correspondence will be weak star upper semicontinuous, and more importantly, will have fixed points.

We will refer to all the assumptions made above concerning spaces and correspondences as [A-1].

2.1 Comments

(1) Given the probability space, (Ω, B_Ω, μ), metric spaces, (Z, ρ_Z) compact and (X, ρ_X) separable, consider an arbitrary set-valued mapping or a correspondence, Γ, from $\Omega \times Z$ into X taking *nonempty* values in X, denoted

$$\Gamma : \Omega \times Z \longrightarrow P(X). \tag{8}$$

For any metric space (X, ρ_X), $P(X)$ will denote the collection of all nonempty subsets of X, and $P_f(X) := P_{\rho_X f}(X)$ will denote the collection of all nonempty and ρ_X-closed subsets of X (we will often leave off the subscript denoting the metric). Given ω and z, we have for any subset S of X the following definitions,

$$\left.\begin{array}{c} \Gamma_\omega^-(S) := \{z \in Z : \Gamma_\omega(z) \cap S \neq \varnothing\}, \\ \text{and} \\ \Gamma_z^-(S) := \{\omega \in \Omega : \Gamma_z(\omega) \cap S \neq \varnothing\}, \end{array}\right\} \tag{9}$$

where for fixed ω, $\Gamma_\omega(\cdot) := \Gamma(\omega, \cdot)$, and for fixed z, $\Gamma_z(\cdot) := \Gamma(\cdot, z)$. Finally, let

$$\Gamma^-(S) := \{(\omega, z) \in \Omega \times Z : \Gamma(\omega, z) \cap S \neq \varnothing\}. \tag{10}$$

Let B_Z and B_X be the Borel σ-fields in Z and X (respectively). We have the following definitions. Given correspondence, $\Gamma(\cdot,\cdot)$, we say that,

(a) $\Gamma_z(\cdot)$ is weakly measurable (or measurable) if for all S open in X, $\Gamma_z^-(S) \in B_\Omega$,
(b) $\Gamma_\omega(\cdot)$ is upper semicontinuous if for all S closed X, $\Gamma_\omega^-(S)$ is ρ_Z-closed,
(c) $\Gamma(\cdot,\cdot)$ is product measurable if for all S open in X, $\Gamma^-(S) \in B_\Omega \times B_Z$.
(d) $\Gamma(\cdot,\cdot)$ is upper Caratheodory if $\Gamma(\cdot,\cdot)$ is product measurable and for each ω, $\Gamma_\omega(\cdot)$ is upper semicontinuous.

For X a separable metric space, weak measurability of $\Gamma_z(\cdot)$ implies that for each z,

$$Gr\Gamma_z(\cdot) := \{(\omega, x) \in \Omega \times X : x \in \Gamma_z(\omega)\} \in B_\Omega \times B_X. \tag{11}$$

Finally, for X compact and $\Gamma(\cdot,\cdot)$ upper Caratheodory, we have by Lemma 3.1 in Kucia and Nowak (2000) that the mapping

$$\omega \longrightarrow Gr\Gamma_{\omega}(\cdot) \in P_f(Z \times X) \tag{12}$$

is measurable - i.e., for S an open subset of $Z \times X$, $(Gr\Gamma_{(\cdot)}(\cdot))^{-}(S) \in B_{\Omega}$, where

$$(Gr\Gamma_{(\cdot)}(\cdot))^{-}(S) := \{\omega \in \Omega : Gr\Gamma_{\omega}(\cdot) \cap S \neq \varnothing\}. \tag{13}$$

(2) Let (Z, ρ_Z) be any metric space. Consider the hyperspace of nonempty, ρ_Z-closed subsets of Z, $P_f(Z)$. The distance from a point $z \in Z$ to a set $C \in P_f(Z)$ is given by

$$dist(z, C) := \inf_{z' \in C} \rho_Z(z, z'). \tag{14}$$

Given two sets B and C in $P_f(Z)$, the excess of B over C is given by

$$e_{\rho_Z}(B, C) := \sup_{z \in B} dist_{\rho_Z}(z, C). \tag{15}$$

The given two sets B and C in $P_f(Z)$, the Hausdorff distance in $P_f(Z)$ between B and C is given by

$$h_{\rho_Z}(B, C) = \max\{e_{\rho_Z}(B, C), e_{\rho_Z}(C, B)\}. \tag{16}$$

If (Z, ρ_Z) is separable, then $(P_f(Z), h_{\rho_Z})$ is a separable metric space. If (Z, ρ_Z) is compact, then $(P_f(Z), h_{\rho_Z})$ is a compact metric space (see Aliprantis and Border, 2006). Often we will write h rather than h_{ρ_Z} - when the underlying metric is clear.

(3) Again let (Z, ρ_Z) be any metric space. Z is said to be connected if it cannot be written as the union of two nonempty, disjoint open subsets of Z. Equivalently, Z is connected if and only if the only subsets of Z that are open and closed in Z are the empty set and Z itself. If Z is compact and connected it is called a continuum.

2.2 w^*-Convergence and K-Convergnece in $\mathcal{L}_Y^{\infty}$

A sequence, $\{v^n\}_n \subset \mathcal{L}_Y^{\infty}$, converges weak star to $v^* = (v_1^*(\cdot), \ldots, v_m^*(\cdot)) \in \mathcal{L}_Y^{\infty}$, denoted by $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$, if and only if

$$\int_{\Omega} \langle v^n(\omega), l(\omega)\rangle_{R^m}\, d\mu(\omega) \longrightarrow \int_{\Omega} \langle v^*(\omega), l(\omega)\rangle_{R^m}\, d\mu(\omega) \tag{17}$$

for all $l(\cdot) \in \mathcal{L}^1_{R^m}$.

A sequence, $\{v^n\}_n \subset \mathcal{L}_Y^{\infty}$, K-convergences (i.e., Komlos convergence - Komlos, 1967) to $\widehat{v} \in \mathcal{L}_Y^{\infty}$, denoted by $v^n \underset{K}{\longrightarrow} \widehat{v}$, if and only if every subsequence, $\{v^{n_k}(\cdot)\}_k$, of $\{v^n(\cdot)\}_n$ has an arithmetic mean sequence, $\{\widehat{v}^{n_k}(\cdot)\}_k$, where

$$\widehat{v}^{n_k}(\cdot) := \frac{1}{k}\sum_{q=1}^{k} v^{n_q}(\cdot), \tag{18}$$

such that

$$\widehat{v}^{n_k}(\omega) \underset{R^m}{\longrightarrow} \widehat{v}(\omega) \;\; a.e. \;\; [\mu]. \tag{19}$$

The relationship between w^*-convergence and K-convergence is summarized via the following results which follow from Balder (2000): For every sequence of value functions,

$\{v^n\}_n \subset \mathcal{L}_Y^\infty$, and $\widehat{v} \in \mathcal{L}_Y^\infty$ the following statements are true:

$$\left.\begin{array}{c}\text{(i) If the sequence } \{v^n\}_n \ K\text{-converges to } \widehat{v}, \text{ then } \{v^n\}_n \ w^*\text{-converges to } \widehat{v}. \\ \\ \text{(ii) The sequence } \{v^n\}_n \ w^*\text{-converges to } \widehat{v} \text{ if and only if} \\ \text{every subsequence } \{v^{n_k}\}_k \text{ of } \{v^n\}_n \text{ has a further subsequence, } \{v^{n_{k_r}}\}_r, \\ K\text{-converging to } \widehat{v}.\end{array}\right\} \tag{20}$$

For any sequence of value function profiles, $\{v^n\}_n$, in $\mathcal{L}_Y^\infty$ it is automatic that

$$sup_n \int_\Omega \|v^n(\omega)\|_{R^m} \, d\mu(\omega) < +\infty. \tag{21}$$

Thus, by the classical Komlos Theorem (1967), any such sequence, $\{v^n\}_n$, has a subsequence, $\{v^{n_k}\}_k$ that K-converges to some K-limit, $\widehat{v} \in \mathcal{L}_Y^\infty$.

3 USCOs and Upper Caratheodory Correspondences

3.1 USCOs

We have compact metric spaces $(\mathcal{L}_Y^\infty, \rho_{w^*})$ and (X, ρ_X). Let $\mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)} := \mathcal{U}(\mathcal{L}_Y^\infty, P_f(X))$ denote the collection of all upper semicontinuous correspondences taking nonempty, ρ_X-closed (and hence ρ_X-compact) values in X. Following the literature, we will call such mappings, USCOs (see Crannell, Franz, and LeMasurier, 2005, Anguelov and Kalenda, 2009, and Hola and Holy, 2009). Given any $\mathcal{N} \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$, denote by $\mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}[\mathcal{N}]$ the collection of all sub-USCOs belonging to N, that is, all USCOs $\phi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$ whose graph,

$$Gr\phi := \{(v, x) \in \mathcal{L}_Y^\infty \times X : x \in \phi(v)\},$$

is contained in the graph of $\mathcal{N}$,

$$Gr\mathcal{N} := \{(v, x) \in \mathcal{L}_X^\infty \times X : x \in \mathcal{N}(v)\}.$$

We will call any sub-USCO, $\phi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}[\mathcal{N}]$ a minimal USCO belonging to $\mathcal{N}$, if for any other sub-USCO, $\psi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}[\mathcal{N}]$, $Gr\psi \subseteq Gr\phi$ implies that $Gr\psi = Gr\phi$ (see Drewnowski and Labuda, 1990). We will use the special notation, $[\mathcal{N}]$, to denote the collection of all minimal USCOs belonging to $\mathcal{N}$.

3.2 Upper Caratheodory Sub-Correspondences

Consider the uC correspondence $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$, and let

$$\mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}P_f(X)}[\mathcal{N}(\cdot,\cdot)] := \mathcal{UC}^{\mathcal{N}} \tag{22}$$

denote the collection of all upper Caratheodory mappings belonging to $\mathcal{N}(\cdot,\cdot)$. Thus, $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$ if and only if $\eta(\cdot,\cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}P_f(X)}$ and

$$Gr\eta(\omega,\cdot) \subset Gr\mathcal{N}(\omega,\cdot) \text{ for all } \omega.$$

We will refer to the uC correspondence $\eta(\cdot,\cdot)$ as a uC sub-correspondence belonging to $\mathcal{N}(\cdot,\cdot)$.

3.3 Connectedness and Caratheodory Approximability

Consider the uC composition correspondence,

$$\left.\begin{array}{r}(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := u(\omega, v, \mathcal{N}(\omega, v)) \\ := (u_1(\omega, v, \mathcal{N}(\omega, v)), \ldots, u_m(\omega, v, \mathcal{N}(\omega, v))) \in P_f(Y).\end{array}\right\} \tag{23}$$

where $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty \text{-} P_f(X)}$ and the function,

$$(\omega, v, x) \longrightarrow u(\omega, v, x) := (u_1(\omega, v, x), \ldots, u_m(\omega, v, x)) \in Y, \tag{24}$$

is Caratheodory, measurable in ω and jointly continuous in (v, x). For all uC sub-correspondences, $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$ the induced sub-correspondence

$$p(\omega, v) := u(\omega, v, \eta(\omega, v)) := (\underbrace{u_1(\omega, v, \eta(\omega, v))}_{p_1(\omega,v)}, \ldots, \underbrace{u_m(\omega, v, \eta(\omega, v))}_{p_m(\omega,v)}) \in P_f(Y), \tag{25}$$

is a uC sub-correspondence belonging to $\mathcal{P}(\cdot,\cdot)$. Thus, $p(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{P}}$. Each uC sub-correspondence in $\mathcal{UC}^{\mathcal{P}}$ induces a selection sub-correspondence, $v \longrightarrow \mathcal{S}^\infty(p(\cdot, v)) := \mathcal{S}^\infty(p_1(\cdot, v)) \times \cdots \times \mathcal{S}^\infty(p_m(\cdot, v))$, and we will show that if the underlying uC sub-correspondence, $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$, is continuum valued then this selection sub-correspondence is weak star upper semicontinuous in v and has fixed points. Thus, we will show that there exists $v^* \in \mathcal{L}_Y^\infty$, such that

$$v^* \in \mathcal{S}^\infty(p(\cdot, v^*)) \subset \mathcal{S}^\infty(\mathcal{P}(\cdot, v^*)) \subset \mathcal{L}_Y^\infty. \tag{26}$$

For $d = 1, 2, \ldots, m$, consider the uC sub-correspondence,

$$(\omega, v) \longrightarrow p_d(\omega, v) := u_d(\omega, v, \eta(\omega, v)) \subset \mathcal{P}_d(\omega, v) \in P_f(Y_d). \tag{27}$$

Definitions 1 *(Caratheodory Approximable uC Correspondences)*
We say that $p_d(\cdot,\cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty \text{-} P_f(Y_d)}$ *is Caratheodory approximable if for each* $\varepsilon > 0$ *there is a Caratheodory function,* $g_d^\varepsilon(\cdot,\cdot) : \Omega \times \mathcal{L}_Y^\infty \longrightarrow Y_d$*, having the property that for each* $(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty$ *and each* $(v, g_d^\varepsilon(\omega, v)) \in \mathcal{L}_Y^\infty \times Y_d$ *there exists* $(\overline{v}^d, \overline{u}_d) \in Grp_d(\omega, \cdot)$ *such that*

$$\rho_{w^*}(v, \overline{v}^d) + \rho_{Y_d}(g_d^\varepsilon(\omega, v), \overline{u}_d) < \varepsilon. \tag{28}$$

We call this Caratheodory function, $g^\varepsilon(\cdot,\cdot)$*, an* ε*-Caratheodory selection of* $p_d(\cdot,\cdot)$ *- or equivalently, a Caratheodory function,* $g_d^\varepsilon : \Omega \times \mathcal{L}_Y^\infty \longrightarrow Y_d$*, such that for each* ω

$$Grg_d^\varepsilon(\omega, \cdot) \subset B_{\rho_{w^*\times Y_d}}(\varepsilon, Grp_d(\omega, \cdot)). \tag{29}$$

We say that the uC correspondence, $\mathcal{P}_d(\cdot,\cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty \text{-} P_f(Y_d)}$*, is Caratheodory approximable if* $\mathcal{P}(\cdot,\cdot)$ *has a uC sub-correspondence,* $p_d(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{P}}$*, such that for each* $\varepsilon > 0$*,* $p_d(\cdot,\cdot)$ *has an* ε*-Caratheodory Selection.*

By Corollary 4.3 in Kucia and Nowak (2000), a sufficient condition for $p_d(\cdot,\cdot)$ to be Caratheodory approximable, and therefore, for $p_d(\cdot,\cdot)$ to have for each $\varepsilon > 0$ an ε-Caratheodory selection, is for the uC sub-correspondence, $p_d(\cdot,\cdot)$, to have closed, interval values.

4 A Fixed Point Theorem for Measurable Selection Valued Correspondences Induced by uC Composition Correspondences

We will show here, under assumptions [A-1], that for any uC composition correspondence,

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := u(\omega, v, \mathcal{N}(\omega, v)), \tag{30}$$

if there exists a uC sub-correspondence, $\eta(\cdot, \cdot) \in \mathcal{UC}^{\mathcal{N}}$, taking *continuum values* in X (*closed and connected values* in X), then for each $d = 1, 2, \ldots, m$, the uC composition sub-correspondence,

$$(\omega, v) \longrightarrow p_d(\omega, v) := u_d(\omega, v, \eta(\omega, v)), \tag{31}$$

takes closed, interval values in Y_d, and therefore, by Corollary 4.3 in Kucia and Nowak (2000), $p_d(\cdot, \cdot)$ is Caratheodory approximable. As a consequence, we are able to show that there exists a function $v^* \in \mathcal{L}_Y^{\infty}$ such that

$$v^*(\omega) \in \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu],$$

or equivalently,

$$v^* \in \mathcal{S}^{\infty}(\mathcal{P}_{v^*}).$$

Here is our main result.

Theorem *(A selection correspondence induced by a uC composition correspondence with underlying continuum valued uC correspondence has fixed points)*
Suppose assumptions [A-1] hold. Let

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := u(\omega, v, \mathcal{N}(\omega, v))$$

be a uC composition correspondence where $\mathcal{N}(\cdot, \cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^{\infty}\text{-}P_f(X)}$ *and* $(\omega, v, x) \longrightarrow u(\omega, v, x) \in Y$ *is Caratheodory. If the uC correspondence,* $\mathcal{N}(\cdot, \cdot)$, *contains a uC sub-correspondence,* $\eta(\cdot, \cdot)$, *taking closed connected values in* X, *then there exists* $\widehat{v} \in \mathcal{L}_Y^{\infty}$ *such that*

$$\widehat{v}(\omega) \in \mathcal{P}(\omega, \widehat{v}) \text{ a.e. } [\mu].$$

Proof: As noted above, because $\eta(\cdot, \cdot)$ takes closed and connected values, the induced uC composition sub-correspondence,

$$\left.\begin{array}{c} (\omega, v) \longrightarrow p(\omega, v) := (p_1(\omega, v), \ldots, p_m(\omega, v)) \\ = (u_1(\omega, v, \eta(\omega, v)), \ldots, u_m(\omega, v, \eta(\omega, v))) := u(\omega, v, \eta(\omega, v)), \end{array}\right\} \tag{32}$$

is such that for each $d = 1, 2, \ldots, m$, $(\omega, v) \longrightarrow p_d(\omega, v)$, takes closed interval values in Y_d, implying via Corollary 4.3 in Kucia and Nowak (2000) that $p_d(\cdot, \cdot)$ is Caratheodory approximable. Thus, there is a sequence of m-tuples of Caratheodory functions,

$$\{g^n(\cdot, \cdot)\}_n := \{(g_1^n(\cdot, \cdot), \ldots, g_m^n(\cdot, \cdot))\}_n, \tag{33}$$

such that for each n and for each $(\omega, v) \in \Omega \times \mathcal{L}_Y^{\infty}$ there exists for each d, $(\overline{v}^{nd}, \overline{u}_d^n) \in Grp_d(\omega, \cdot)$ such that,

$$\rho_{w^*}(v, \overline{v}^{nd}) + \rho_{Y_d}(g_d^n(\omega, v), \overline{u}_d^n) < \tfrac{1}{m \cdot n}. \tag{34}$$

Next, consider the mapping from $\mathcal{L}_Y^\infty$ to $\mathcal{L}_Y^\infty$ given by

$$v \longrightarrow T^n(v) := g^n(\cdot, v) := (g_1^n(\cdot, v), \ldots, g_m^n(\cdot, v)) \in \mathcal{L}_Y^\infty. \tag{35}$$

Observe that for each n, $T^n(\cdot)$ is continuous (i.e., $v^k \underset{\rho_{w^*}}{\longrightarrow} v^*$ implies that $T^n(v^k) \underset{\rho_{w^*}}{\longrightarrow} T^n(v^*)$). This is true because for each n, $v^k \underset{\rho_{w^*}}{\longrightarrow} v^*$ implies that for each $\omega \in \Omega$, as $k \longrightarrow \infty$, $g^n(\omega, v^k) \underset{\rho_Y}{\longrightarrow} g^n(\omega, v^*) \in Y$. Therefore, for $l \in \mathcal{L}_{R^m}^1$ chosen arbitrarily, $\left\langle g^n(\omega, v^k), l(\omega)\right\rangle \underset{R}{\longrightarrow} \langle g^n(\omega, v^*), l(\omega)\rangle$ a.e. $[\mu]$, implying that as $k \longrightarrow \infty$,

$$\int_\Omega \left\langle g^n(\omega, v^k), l(\omega)\right\rangle d\mu(\omega) \longrightarrow \int_\Omega \langle g^n(\omega, v^*), l(\omega)\rangle \, d\mu(\omega).$$

Since the choice of $l \in \mathcal{L}_{R^m}^1$ was arbitrary, we can conclude that if $v^k \underset{\rho_{w^*}}{\longrightarrow} v^*$, then $g^n(\cdot, v^k) \underset{\rho_{w^*}}{\longrightarrow} g^n(\cdot, v^*) \in \mathcal{L}_Y^\infty$. By the Brouwer-Schauder-Tychonoff Fixed Point Theorem (e.g., see Aliprantis-Border, 17.56, 2006), for each n, there exists $v^n \in \mathcal{L}_Y^\infty$ such that

$$v^n = T^n(v^n) := g^n(\cdot, v^n). \tag{36}$$

Thus, we have for each n a set, N^n, of μ-measure zero such that

$$v^n(\omega) = g^n(\omega, v^n) \text{ for all } \omega \in \Omega \backslash N^n,\ \mu(N^n) = 0. \tag{37}$$

Letting $N^\infty := \cup_n N^n$ - so that, $\mu(N^\infty) = 0$ - we have for each $n = 1, 2, \ldots$and for each $d = 1, 2, \ldots, m$, that

$$v_d^n(\omega) = g_d^n(\omega, v^n) \text{ for all } \omega \in \Omega \backslash N^\infty,\ \mu(N^\infty) = 0. \tag{38}$$

Call the equation (38), one for each n, the Caratheodory equation and call the sequence, $\{v^n\}_n$, in $\mathcal{L}_Y^\infty$ the *Caratheodory fixed point sequence*.

For each pair of m-tuples of Caratheodory approximating functions and fixed points, $(g^n(\cdot, \cdot), v^n)$, consider the measurable function,

$$\omega \longrightarrow \min\nolimits_{(v, u_d) \in Grp_d(\omega, \cdot)} [\rho_{w^*}(v^n, v) + \rho_{Y_d}(g_d^n(\omega, v^n), u_d)], \tag{39}$$

By Lemma 3.1 in Kucia and Nowak (2000) the graph correspondence, $\omega \longrightarrow Grp_d(\omega, \cdot)$, is measurable, and therefore, by the continuity of the function

$$(v, u_d) \longrightarrow [\rho_{w^*}(v^n, v) + \rho_{Y_d}(g_d^n(\omega, v^n), u_d)]$$

on $\mathcal{L}_Y^\infty \times Y_d$, there exists for each n, a measurable (everywhere) selection of $Grp_d(\omega, \cdot)$,

$$\omega \longrightarrow (\overline{v}_\omega^{nd}, \overline{u}_{\omega d}^n) \in \mathcal{L}_Y^\infty \times Y_d \tag{40}$$

solving the minimization problem (39) state-by-state (see Himmelberg, Parthasarathy, and VanVleck, 1976). Moreover, we have by the Caratheodory approximability of uC Nash payoff sub-correspondence,

$$p(\cdot, \cdot) := (p_1(\cdot, \cdot), \ldots, p_m(\cdot, \cdot)),$$

and (34) above that for the sequences of optimal selections, $\{(\overline{v}_{(\cdot)}^{nd}, \overline{u}_{(\cdot)d}^n)\}_n$, $d = 1, 2, \ldots, m$, where for each n and for each ω, $\overline{v}_\omega^{nd} \in \mathcal{L}_Y^\infty$ and $\overline{u}_{\omega d}^n \in Y_d$, we have for each n and for each ω,

$$\underbrace{\rho_{w^*}(v^n, \overline{v}_\omega^{nd})}_{A} + \underbrace{\rho_{Y_d}(g_d^n(\omega, v^n), \overline{u}_{\omega d}^n)}_{B} < \tfrac{1}{m \cdot n}. \tag{41}$$

Given (37) and (41), we have for the sequences,

$$\{g^n(\cdot,\cdot), v^n\}_n \text{ and } \{\overline{v}^{nd}_{(\cdot)}, \overline{u}^n_{(\cdot)d}\}_n, d = 1, 2, \ldots, m, \tag{42}$$

that for all $\omega \in \Omega \backslash N^\infty$, $\mu(N^\infty) = 0$, and for all n,

$$\rho_{w^*}(v^n, \overline{v}^{nd}_\omega) + \underbrace{\rho_{Y_d}(v^n_d(\omega), \overline{u}^n_{\omega d})}_{C} < \tfrac{1}{m \cdot n}, \tag{43}$$

where for each d and for each n, $\omega \longrightarrow \overline{v}^{nd}_\omega$ is $\mathcal{L}^\infty_Y$-valued, while $\omega \longrightarrow \overline{u}^n_{\omega d}$ is Y_d-valued, and

$$\overline{u}^n_\omega := (\overline{u}^n_{\omega 1}, \ldots, \overline{u}^n_{\omega m}) \in (p_1(\omega, \overline{v}^{n1}_\omega), \ldots, p_m(\omega, \overline{v}^{nm}_\omega)) \text{ for all } \omega \in \Omega. \tag{44}$$

Next, because $(\mathcal{L}^\infty_Y, \rho_{w^*})$ is a compact metric space we can assume without loss of generality that the sequence of fixed points in $\mathcal{L}^\infty_Y$, $\{v^n\}_n$, K-converges to some $\widehat{v} \in \mathcal{L}^\infty_Y$, implying that $v^n \underset{\rho_{w^*}}{\longrightarrow} \widehat{v}$ and therefore implying via (41)A that $\overline{v}^{nd}_\omega \underset{\rho_{w^*}}{\longrightarrow} \widehat{v}$ uniformly in d and ω. Moreover, by (43)C, we have that

$$\widehat{\overline{u}}^n_{\omega d} = \tfrac{1}{n} \sum_{k=1}^n \overline{u}^k_{\omega d} \underset{\rho_{Y_d}}{\longrightarrow} \widehat{v}_d(\omega) \text{ a.e. } [\mu], \tag{45}$$

where for each n, $\overline{u}^n_{\omega d} \in p_d(\omega, \overline{v}^{nd}_\omega)$ for all ω. By the properties of K-convergence, for each $n = 1, 2, 3, \ldots$, there is a set, $\widehat{N}^n$, of μ-measure zero such that for all d and for all $\omega \in \Omega \backslash \widehat{N}^n$ as $q \longrightarrow \infty$

$$\left.\begin{array}{c} \widehat{\overline{u}}^{n+q}_{\omega d} = \tfrac{1}{q} \sum_{r=1}^q \overline{u}^{n+r}_{\omega d} \underset{\rho_{Y_d}}{\longrightarrow} \widehat{v}_d(\omega), \\ \text{and} \\ \widehat{v}^{n+q}_d(\omega) = \tfrac{1}{q} \sum_{r=1}^q v^{n+r}_d(\omega) \underset{\rho_{Y_d}}{\longrightarrow} \widehat{v}_d(\omega). \end{array}\right\} \tag{46}$$

Letting $\widehat{N}^\infty := \cup_{n=1}^\infty \widehat{N}^n$ we have that for any $n = 1, 2, 3, \ldots$, that for each player the truncated sequences, $\{\overline{u}^{n+q}_{(\cdot)d}\}_{q=1}^\infty$ and $\{v^{n+q}_d(\cdot)\}_{q=1}^\infty$, have arithmetic mean sequences, $\{\widehat{\overline{u}}^{n+q}_{(\cdot)d}\}_{q=1}^\infty$ and $\{\widehat{v}^{n+q}_d(\cdot)\}_{q=1}^\infty$, converging pointwise to $\widehat{v}_d(\cdot)$ off the set $\widehat{N}^\infty$ of μ-measure zero where the exceptional set $\widehat{N}^\infty$ is independent of n.

Because $p_d(\omega, \cdot)$ is ρ_{w^*}-ρ_{Y_d}-upper semicontinuous and because for each d, $\overline{v}^{nd}_\omega \underset{\rho_{w^*}}{\longrightarrow} \widehat{v}$ uniformly in d and ω, we have for each d and ω and for any sequence of $k_\omega = 1, 2, \ldots$, increasing to ∞, that there is a sequence $\{n_{k_\omega}\}_{k_\omega}$ increasing to ∞, such that for all $n \geq n_{k_\omega}$ the ρ_{Y_d}-open ball, $B_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, \widehat{v}))$, about $p_d(\omega, \widehat{v})$ of radius $\frac{1}{k_\omega}$ with closure given by the closed, convex ball, $\overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, \widehat{v}))$, about $p_d(\omega, \widehat{v})$ of radius $\frac{1}{k_\omega}$, is such that for all $n \geq n_{k_\omega}$ and $q = 1, 2, \ldots$.

$$p_d(\omega, \overline{v}^{(n+q)d}_\omega) \subset B_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, \widehat{v})) \subset \overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, \widehat{v})). \tag{47}$$

Moreover, for all $\omega \in \Omega \backslash (N^\infty \cup \widehat{N}^\infty)$, $n \geq n_{k_\omega}$, and $q = 1, 2, \ldots$, we have for each d

$$\overline{u}^{n+q}_{\omega d} \in p_d(\omega, \overline{v}^{(n+q)d}_\omega) \subset \overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, \widehat{v})). \tag{48}$$

Because $\overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, \widehat{v}))$ is closed and convex, and because

$$\widehat{\overline{u}}_{\omega d}^{n+q} \in \overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, \widehat{v})) \text{ for all } \omega \in \Omega \backslash (N^\infty \cup \widehat{N}^\infty),\ n \geq n_{k_\omega}, \text{ and } q = 1, 2, \ldots, \tag{49}$$

the fact that for each d, $\widehat{\overline{u}}_{\omega d}^{n+q} \underset{\rho_{Y_d}}{\longrightarrow} \widehat{v}_d(\omega)$ for each $\omega \in \Omega \backslash (N^\infty \cup \widehat{N}^\infty)$ and for each $n \geq n_{k_\omega}$ as $q = 1, 2, \ldots$, goes to ∞, implies that for each d and for all $\omega \in \Omega \backslash (N^\infty \cup \widehat{N}^\infty)$,

$$\widehat{v}_d(\omega) \in \overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, \widehat{v})) \text{ for all } k_\omega. \tag{50}$$

Thus, as $k_\omega \longrightarrow \infty$ we have in the limit for each d and for each $\omega \in \Omega \backslash (N^\infty \cup \widehat{N}^\infty)$

$$\widehat{v}_d(\omega) \in p_d(\omega, \widehat{v}).$$

Thus, we have $\widehat{v} = (\widehat{v}_1, \ldots, \widehat{v}_m)$ such that

$$\widehat{v}(\omega) \in p(\omega, \widehat{v}) \subset \mathcal{P}(\omega, \widehat{v}) \text{ a.e. } [\mu]. \tag{51}$$

Q.E.D.

5 Comments

(1) Note that, due to the fact that Komlos convergence implies weak star convergence, the arguments given in the latter part of the proof above (see expressions (45)-(50) above) establish that the uC Nash payoff sub-correspondence induces a weak star upper semicontinuous selection sub-correspondence, $v \longrightarrow \mathcal{S}^\infty(p_v)$.

(2) Fu and Page (2022a) established that all $\mathcal{PSG}s$ satisfying assumptions [A-1] above have uC Nash correspondences given by a bundle of minimal uC Nash correspondences each of which takes minimally essential, connected Nash values. Given that all $\mathcal{DSG}s$ satisfying the usual assumptions have one-shot games satisfying assumptions [A-1], all such $\mathcal{DSG}s$ have Nash payoff selection correspondences having fixed points - implying that all such $\mathcal{DSG}s$ have stationary Markov perfect equilibria (SMPE) - see Fu and Page (2022b).

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